Root system chip-firing

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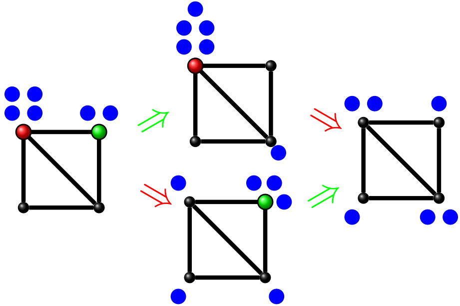
Joint work with Pavel Galashin, Thomas McConville, and Alexander Postnikov

Classical chip-firing

Classical chip-firing (as introduced by Björner-Lovász-Shor, 1991) is a discrete dynamical system that takes place on a graph. The states are configurations of chips on the vertices. We may *fire* a vertex that has at least as many chips as neighbors, sending one chip to each neighbor:



A key property of this system is that it is *confluent*: from a given initial configuration, either all sequences of firings go on forever, or they all terminate at the same *stable* configuration (called the *stabilization*).

Chip-firing on a line

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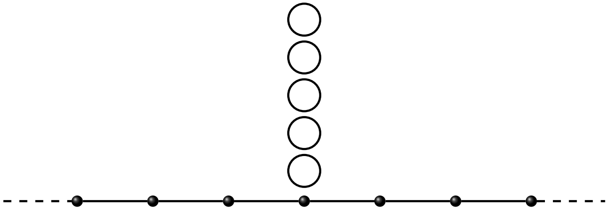

Even in this simple setting we can ask questions such as: which stable configuration does n chips at the origin stabilize to?

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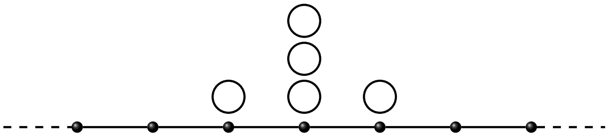

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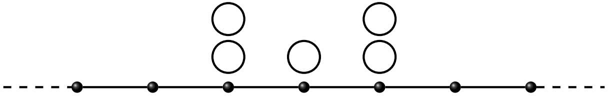

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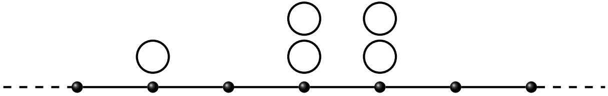

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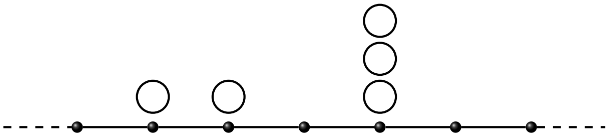

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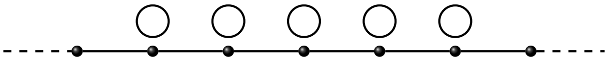

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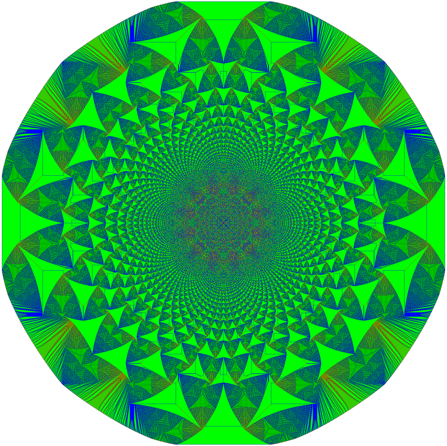
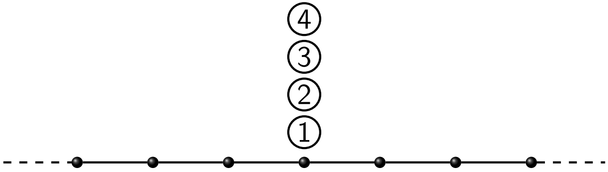
A brief aside...

Things are a bit different in two dimensions (studied by physicists since Bak-Tang-Wiesenfeld, 1987). Stabilization of 4×10^6 chips at origin of $\mathbb{Z}^2$:



Labeled chip-firing

Jim Propp recently introduced a “labeled” variant of chip-firing (on $\mathbb{Z}$). All the chips are now *distinguishable*, given labels from $\mathbb{N}$. Whenever two chips occupy the same vertex we can fire them together, moving the lesser-labeled chip leftwards and the greater-labeled chip rightwards:



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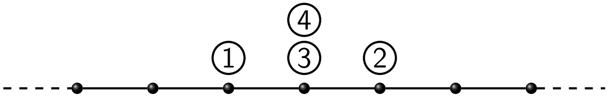

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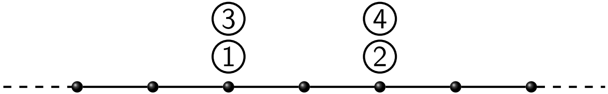

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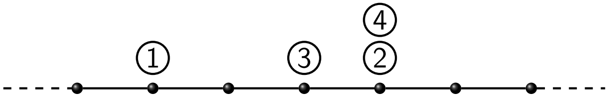

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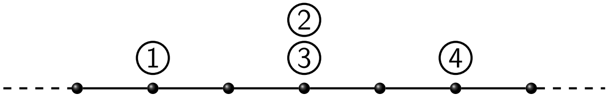

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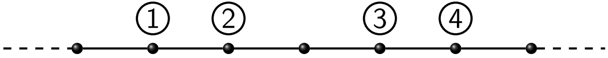
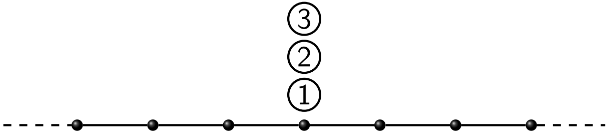


Labeled chip-firing is not confluent in general

What if we started with three chips at the origin:



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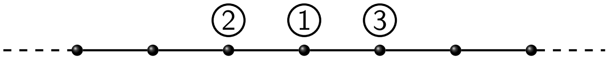
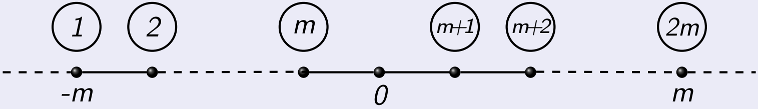


Not sorted and not confluent!

Sorting an even number of chips

Theorem (H.-McConville-Propp, 2017)

Suppose $n = 2m$ is even. Then starting from n labeled chips at the origin, the chip-firing process "sorts" the chips to a unique stable configuration:



Root system reformulation of labeled chip-firing

For any configuration of n labeled chips, if we set $c := (c_1, \ldots, c_n) \in \mathbb{Z}^n$ where

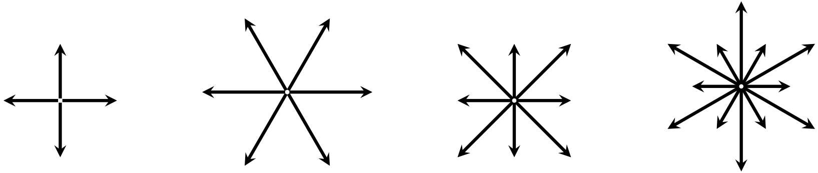
$$c_i := \text{the position of the chip } \textcircled{i},$$

then, for $1 \leq i < j \leq n$, we are allowed to fire chips $\textcircled{i}$ and $\textcircled{j}$ in this configuration as long as c is orthogonal to $e_j - e_i$; and doing so replaces the vector c by $c + (e_j - e_i)$.

Key observation: the vectors $e_j - e_i$ for $1 \leq i < j \leq n$ are exactly the positive roots Φ^+ of the root system Φ of Type A_{n-1}.

Root system basics

Root systems are certain highly symmetrical finite sets of vectors in Euclidean space. The key property of a root system Φ is that it's preserved by the reflection s_α across the hyperplane orthogonal to any root $\alpha \in \Phi$.



$A_1 \times A_1$ A_2 B_2 G_2

Root systems were first introduced in the late 19th century in the context of Lie theory because they correspond bijectively to simple Lie algebras. They have been classified into *Cartan-Killing types* A_n, B_n, etc.

Root system notation

Type A is the realm of "classical combinatorics" (e.g., permutations).
Let us go over basic notation for root systems, with Type A as an example:

Object	Notation	Type A_{n-1}
Ambient v.s.	V, $\dim V = r$	$\{v \perp (1,1,\ldots,1) \in \mathbb{R}^n\}$
Roots	$\alpha \in \Phi$	$\pm(e_i - e_j)$, $i < j$
Coroots	$\alpha^\vee = \frac{2}{\langle\alpha,\alpha\rangle}\alpha \in \Phi^\vee$	$\pm(e_i - e_j)$, $i < j$
Positive roots	$\alpha \in \Phi^+$	$e_i - e_j$, $i < j$
Simple roots	$\alpha_1, \ldots, \alpha_r$	$\alpha_i = e_i - e_{i+1}$
Root lattice	$Q = \mathbb{Z}[\Phi]$	$\{v \perp (1,1,\ldots,1) \in \mathbb{Z}^n\}$
Weight lattice	$P = \{v \colon \langle v, \alpha^\vee\rangle \in \mathbb{Z}, \forall \alpha \in \Phi\}$	$\mathbb{Z}^n/(1,1,\ldots,1)$
Fundamental weights	$\omega_1, \ldots, \omega_r$	$\omega_i = (\overbrace{1,\ldots,1}^{i},0,\ldots,0)$
Weyl group	$W = \langle s_\alpha, \alpha \in \Phi\rangle \subseteq GL(V)$	Symmetric group S_n

Central-firing for root systems

The description of labeled chip-firing in terms of positive roots of A_{n-1} generalizes naturally to any root system Φ: for a weight $\lambda \in P$, we allow the firing moves $\lambda \to \lambda + \alpha$ for a positive root $\alpha \in \Phi^+$ whenever λ is orthogonal to α.

We call the resulting system the *central-firing* process for Φ (because we allow firing from a weight λ when λ belongs to the *Coxeter hyperplane arrangement*, which is a central arrangement).

Confluence of central-firing

Question

For any root system Φ and weight $\lambda \in P$, when is central-firing confluent from λ?

Answer: it's complicated.

But it seems somewhat related to the *Weyl vector:*

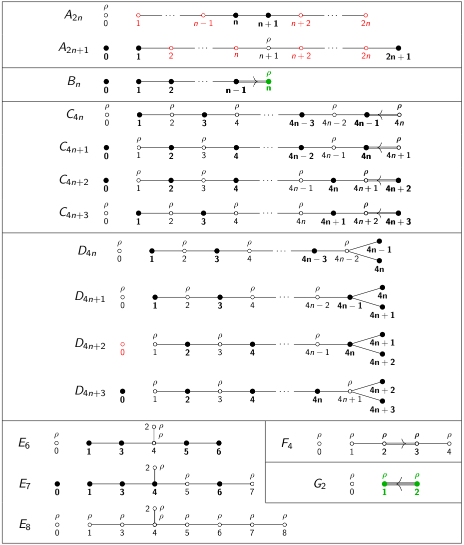
$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{r} \omega_i.$$

Classification of confluence for origin/fundamental weights

Conjecture (G.-H.-M.-P.)

Confluence of central-firing from λ for $\lambda = 0$ or λ a fundamental weight is determined by the table on the right. To first order, central-firing is confluent from λ iff $\lambda \neq \rho$ modulo Q. Exceptions to this are in red or green.

This conjecture is proved in some but not all cases (e.g. for $\lambda = 0$ and $\Phi = A_r$ or B_r, it follows from H.-M.-P. theorems above).



Confluence of central-firing modulo the Weyl group

Theorem (Galashin-H.-McConville-Postnikov)

For any root system Φ, and from any initial weight λ, central-firing is confluent modulo the action of the Weyl group W.

In Type A the Weyl group is the symmetric group, so modding out by the Weyl group is the same as forgetting the labels of chips. Thus this theorem gives a generalization of *unlabeled* chip-firing on a line to any root system.

Note: this is very different from the Cartan matrix chip-firing studied by Benkart-Klivans-Reiner, 2016 (e.g., for $\Phi = A_{n-1}$, ours corresponds to chip-firing of n chips on the infinite path, whereas B.-K.-R. corresponds to chip-firing of any number of chips on the n-cycle).

For *simply laced* root systems can even describe unlabeled central-firing as a certain numbers game on the Dynkin diagram.

Interval-firing

Central-firing has move $\lambda \to \lambda + \alpha$ when $\langle \lambda, \alpha^\vee \rangle = 0$ for $\lambda \in P, \alpha \in \Phi^+$. We found some remarkable deformations of this process.

For any $k \in \mathbb{N}$, define the *symmetric interval-firing process* by

$$\lambda \to \lambda + \alpha \qquad \text{if } \langle \lambda, \alpha^\vee \rangle + 1 \in \{-k, -k+1, \ldots, k-1\}$$

and the *truncated interval-firing process* by

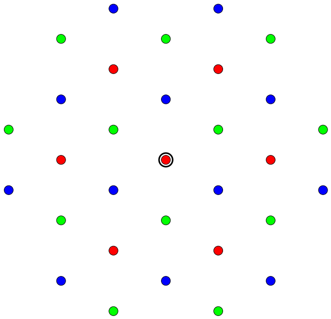
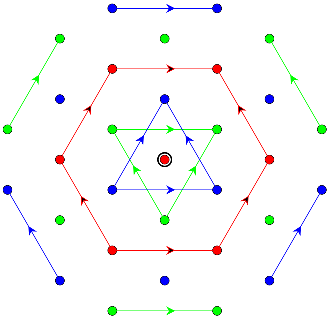
$$\lambda \to \lambda + \alpha \qquad \text{if } \langle \lambda, \alpha^\vee \rangle + 1 \in \{-k+1, -k+2 \ldots, k-1\}.$$

(These are analogous to the extended Φ-Catalan and Φ-Shi hyperplane arrangements, respectively.)

(Taking a particular $k \to \infty$ limit recovers the Cartan matrix chip-firing of Benkart-Klivans-Reiner.)

Pictures of interval-firing

For $\Phi = A_2$:

k	truncated	symmetric
0		

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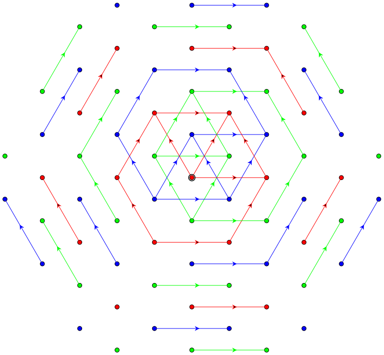
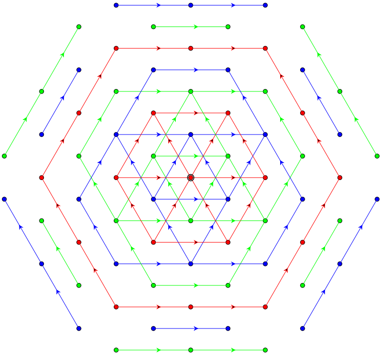
1		

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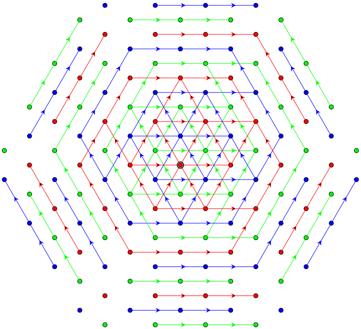
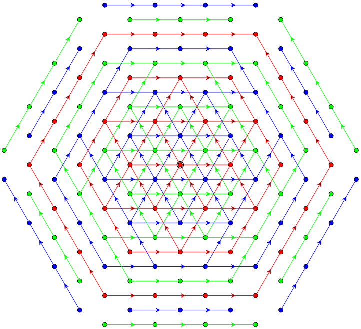
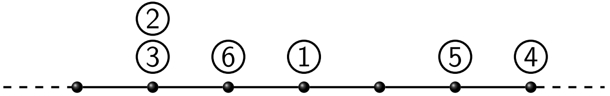
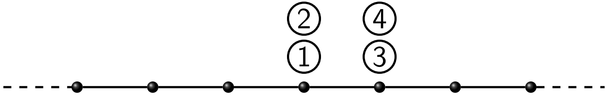
2		

Interval-firing in Type A via chips

When $\Phi = A_{n-1}$, we can interpret interval-firing in terms of chips. The smallest nontrivial case is symmetric $k = 0$ interval-firing, which has $\lambda \to \lambda + \alpha$ for $\lambda \in P, \alpha \in \Phi^+$ when $\langle \lambda, \alpha^\vee \rangle = -1$. This corresponds to allowing *(adjacent) transpositions* of ⓘ and ⓙ if they're out of order:



Here confluence is obvious. The next smallest case is truncated $k = 1$ interval-firing, which has $\lambda \to \lambda + \alpha$ when $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0\}$. This corresponds to allowing transpositions as well as the usual labeled chip-firing moves:



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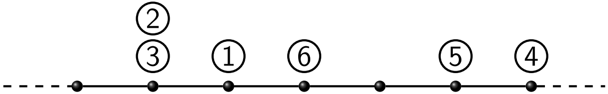

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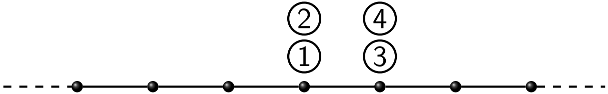

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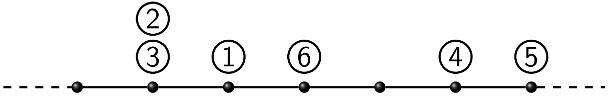

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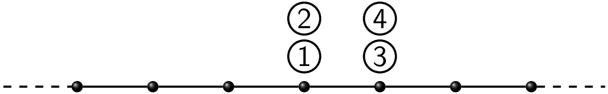

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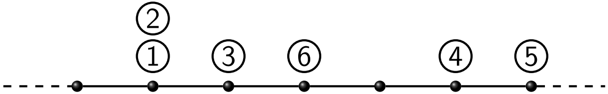

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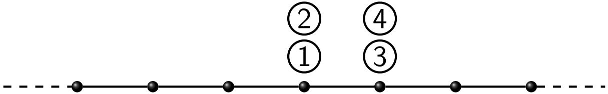

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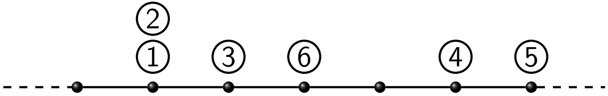

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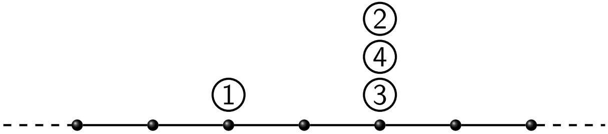

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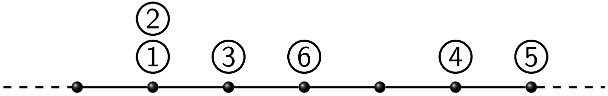

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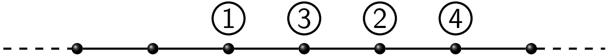
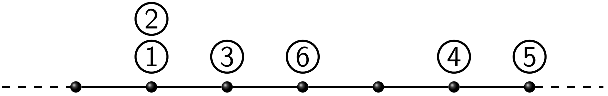
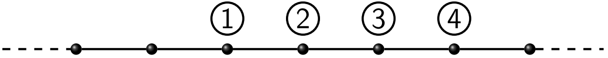


Interval-firing in Type A via chips

When $\Phi = A_{n-1}$, we can interpret interval-firing in terms of chips. The smallest nontrivial case is symmetric $k = 0$ interval-firing, which has $\lambda \to \lambda + \alpha$ for $\lambda \in P, \alpha \in \Phi^+$ when $\langle \lambda, \alpha^\vee \rangle = -1$. This corresponds to allowing *(adjacent) transpositions* of ⓘ and ⓙ if they're out of order:



Here confluence is obvious. The next smallest case is truncated $k = 1$ interval-firing, which has $\lambda \to \lambda + \alpha$ when $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0\}$. This corresponds to allowing transpositions as well as the usual labeled chip-firing moves:



Interval-firing is confluent

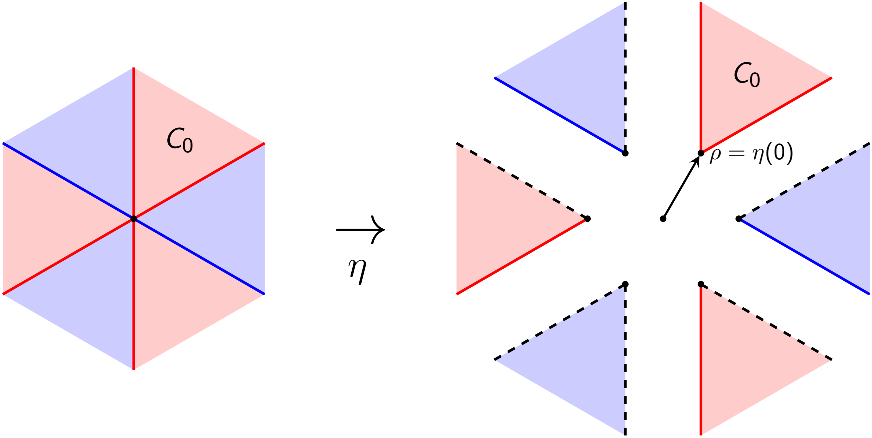
Theorem (Galashin-H.-McConville-Postnikov)

For any root system Φ, and any $k \geq 0$, the symmetric and truncated interval-firing processes are confluent (from all initial points).

The proof is geometric, using the theory of convex polytopes. The main ingredient is a formula for *traverse lengths of permutohedra*.

The map η

Define $\eta\colon P \to P$ by $\eta(\lambda) = \lambda + w_\lambda(\rho)$, where $w_\lambda \in W$ is of min. length such that $w_\lambda^{-1}(\lambda)$ is dominant (so λ dominant $\Rightarrow \eta(\lambda) = \lambda + \rho$).



The stable points of interval-firing

Lemma (Galashin-H.-McConville-Postnikov)

The stable points of symmetric interval-firing are

$$\{\eta^k(\lambda)\colon \lambda \in P, \langle \lambda, \alpha^\vee \rangle \neq -1 \text{ for all } \alpha \in \Phi^+\},$$

and the stable points of truncated interval-firing are

$$\{\eta^k(\lambda)\colon \lambda \in P\}.$$

Ehrhart-like polynomials

In the pictures above, we saw that the set of weights with interval-firing stabilization $\eta^k(\lambda)$ looks "the same" across all values of k, except that it gets "dilated" as k grows.

Following Ehrhart theory, for $k \geq 0$ we define the quantities

$$L_\lambda^{\mathrm{sym}}(k) := \#\mu \text{ with symmetric interval-firing stabilization } \eta^k(\lambda);$$
$$L_\lambda^{\mathrm{tr}}(k) := \#\mu \text{ with truncated interval-firing stabilization } \eta^k(\lambda).$$

Theorem (Galashin-H.-McConville-Postnikov)

For all Φ and all $\lambda \in P$, $L_\lambda^{\mathrm{sym}}(k)$ is a polynomial in k.

Theorem (Galashin-H.-McConville-Postnikov)

For simply laced Φ and all $\lambda \in P$, $L_\lambda^{\mathrm{tr}}(k)$ is a polynomial in k.

Positivity conjectures

Conjecture (Galashin-H.-McConville-Postnikov)

For all Φ and all $\lambda \in P$, both $L^{\mathrm{sym}}_{\lambda}(k)$ and $L^{\mathrm{tr}}_{\lambda}(k)$ are polynomials **with nonnegative integer coefficients** *in k.*

In subsequent work with Alex Postnikov, we proved the "half" of this conjecture concerning $L^{\mathrm{sym}}_{\lambda}(k)$.

Theorem (H.-Postnikov)

Let $\lambda \in P$ be such that $\langle \lambda, \alpha_i^{\vee} \rangle \in \{0,1\}$ for all $1 \leq i \leq r$. Then

$$L^{\mathrm{sym}}_{\lambda}(k) = \sum_{\substack{X \subseteq \Phi^+, \\ \mathit{lin.\ ind.}}} \# \left\{ \mu \in W(\lambda) \colon \begin{array}{c} \langle \mu, \alpha^{\vee} \rangle \in \{0,1\} \text{ for} \\ \text{all } \alpha \in \Phi^+ \cap \mathrm{Span}_{\mathbb{R}}(X) \end{array} \right\} \cdot \mathrm{rVol}(X) \cdot k^{\#X}.$$

Starting point: Stanley's formula (1980) for Ehrhart poly.'s of zonotopes.

Thank you!

References:

- Hopkins, McConville, Propp. "Sorting via chip-firing." Electronic Journal of Combinatorics, 24(3), 2017.
- Galashin, Hopkins, McConville, Postnikov. "Root system chip-firing I: Interval-firing." `arXiv:1708.04850`.
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- Hopkins, Postnikov. "A positive formula for the Ehrhart-like polynomials from root system chip-firing." `arXiv:1803.08472`