

$(\mathrm{GL}_k \times \mathbb{S}_n)$ -MODULES OF MULTIVARIATE DIAGONAL HARMONICS.

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ABSTRACT. This is the first in a series of papers in which we describe explicit structural properties of spaces of diagonal rectangular harmonic polynomials in k sets of n variables, both as GL_k -modules and \mathbb{S}_n -modules, as well as some of their relations to areas such as Algebraic Combinatorics, Representation Theory, Algebraic Geometry, Knot Theory, and Theoretical Physics. Our global aim is to develop a unifying point of view for several areas of research of the last two decades having to do with Macdonald Polynomials Operator Theory, Diagonal Coinvariant Spaces, Rectangular-Catalan Combinatorics, the Delta-Conjecture, Hilbert Scheme of Points in the Plane, Khovanov-Rozansky Homology of (m, n) -Torus links, etc.

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1. INTRODUCTION

Initiated at the beginning of the 1990s, the work of Garsia and Haiman [15] relating to Macdonald polynomials sparked a strong and long-lasting interest in modules of “diagonal harmonic polynomials” in two sets of variables, whose overall dimension is $(n + 1)^{n-1}$. Links to parking functions on the combinatorial side, and Hilbert Schemes of points in the plane [18, 19] on the Algebraic Geometry one, were soon established. These links have become much better understood in recent years. Since its inception, this line of inquiry has opened many fruitful areas of research, including recent ties with Khovanov-Rozansky homology of (m, n) -torus links (see [16, 22, 23, 24, 31]), and ties to the study of super-spaces of Bosons-Fermions. A significant part of this story involves a formula for the (bi)graded character of the above-mentioned modules in the form $\nabla(e_n)$, where ∇ is an operator¹ having the (combinatorial) Macdonald symmetric functions as joint eigenfunctions, here applied to the degree n elementary symmetric function e_n . This operator has many interesting properties on its own, and several important questions about it are still being actively investigated (see [4, 23, 25]).

Our aim in this paper is to describe several new structural properties of the $(\mathrm{GL}_k \times \mathbb{S}_n)$ -module of k -variate diagonal harmonic polynomials², here denoted by $\mathcal{E}_n^{(k)}$, which specialize at $k = 2$ to the module of diagonal harmonic polynomials of Garsia and Haiman. As discussed in [5], there is a filtration of \mathbb{S}_n -module (over the field \mathbb{Q})

$$\mathbb{Q} = \mathcal{E}_n^{(0)} \subset \mathcal{E}_n^{(1)} \subset \mathcal{E}_n^{(2)} \subset \dots \mathcal{E}_{mn}^{(k)} \subset \dots \quad (1.1)$$

which stabilizes³ when k becomes large enough. The first non-trivial term (i.e. when $k = 1$) of this filtration plays a crucial role in several subjects, up to some natural isomorphism. Two striking cases are the cohomology ring of the full flag manifold, and the coinvariant space of the symmetric group \mathbb{S}_n . Both are well-known subjects going back to the 1950s (see [27]). This module $\mathcal{E}_n^{(1)}$ carries a graded version of the regular representation of \mathbb{S}_n , and it may simply be described as the linear span of all derivatives (to all orders) of the Vandermonde determinant in the variables \mathbf{x} :

$$\mathbf{V}(\mathbf{x}) := \prod_{i < j} (x_i - x_j). \quad (1.2)$$

Among the several interesting basis of $\mathcal{E}_n^{(1)}$, maybe the simplest is

$$\{\partial \mathbf{x}^{\mathbf{d}} \mathbf{V}(\mathbf{x}) \mid \mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}, \text{ such that } 0 \leq d_k \leq k - 1\}, \quad (1.3)$$

where

$$\partial \mathbf{x}^{\mathbf{d}} := \frac{\partial^{d_1}}{\partial x_1^{d_1}} \cdots \frac{\partial^{d_n}}{\partial x_n^{d_n}}.$$

This basis highlights that the overall dimension of $\mathcal{E}_n^{(1)}$ is indeed $n!$, and that its graded dimension is the q -analogue of $n!$. The graded irreducible decomposition of $\mathcal{E}_n^{(1)}$ as a \mathbb{S}_n -module is entirely encoded in terms of the Macdonald polynomial $H_n(q; \mathbf{z})$, with $\mathbf{z} = z_1, z_2, \dots$

¹Introduced in [7]. See Appendix 6 for more on this operator.

²Previously discribed in [5]

³In a manner that will be explained further along.

(discussed further in the sequel). Many intriguing properties of this module could certainly be expanded upon, but let us directly skip to the next case with no further ado.

The work of Garsia-Haiman, on the case $k = 2$, is seminal in this whole line of investigation. The corresponding space, denoted by $\mathcal{E}_n^{(2)}$, may be obtained from $\mathcal{E}_n^{(1)}$ by further closing with respect to **higher polarization** operators: $\sum_{i=1}^n y_i^j \frac{\partial}{\partial y_i}$, for any $j \geq 1$ (as well as similar operators obtained by exchanging the role of x and y). Since the early 1990s, many new lines of investigation have been added to their original framework. Noteworthy among recent developments are the successive⁴ appearance: first of “rational”, and then of “rectangular” Catalan combinatorics (see [1, 2, 3, 4]); as well as interesting ties between these combinatorial settings and the elliptic Hall algebra, introduced by Burban-Schiffmann (see [12]). This last context offers a broad extension to the spectrum of operators first considered in [8], particularly those that appear in Theorem 4.4 therein.

In parallel, an explicit study of $\mathcal{E}_n^{(3)}$ (obtained by closing $\mathcal{E}_n^{(2)}$ with respect to polarization operators involving a third set of variables) was started about 10 years ago. Most fundamental questions about it remain open, but its study has suggested new (hard to prove) combinatorial identities linked to the Tamari Lattice as discussed in [9, 10]. The general k -framework is also considered in a previous paper [5], where some broad properties of the modules of k -variate diagonal coinvariant (for any finite complex reflection groups) were established, with the main result as follows. Considering the inductive limit

$$\mathcal{E}_n := \lim_{k \rightarrow \infty} \mathcal{E}_n^{(k)}, \quad (1.4)$$

as a $\mathrm{GL}_\infty \times \mathbb{S}_n$ -module (with commuting actions), an its decomposition into irreducibles:

$$\mathcal{E}_n = \bigoplus_{\mu \vdash n} \bigoplus_{\lambda} (\mathcal{W}_\lambda \otimes \mathcal{V}_\mu)^{\oplus c_{\lambda, \mu}}, \quad (1.5)$$

with the $\{\mathcal{W}_\lambda\}_\lambda$'s representatives of polynomial irreducible representations of GL_∞ , and the $\{\mathcal{V}_\lambda\}_\lambda$'s representatives of irreducible representations of \mathbb{S}_n ; it is shown in [5] that the partitions λ 's, for which the multiplicities $c_{\lambda, \mu}$ do not vanish, have at most $n - 1$ parts and are of size (sum of parts) at most $\binom{n}{2}$. The “character” of this stable limit will be written in the form

$$\mathcal{E}_n = \sum_{\mu \vdash n} \sum_{\lambda} c_{\lambda, \mu} s_\lambda \otimes s_\mu, \quad (1.6)$$

with the Schur functions s_λ 's characters of irreducible polynomial representations of GL_∞ , and the Schur functions s_μ Frobenius transforms of irreducible \mathbb{S}_n -representations. Since the action of GL_∞ on \mathcal{E}_n commutes with that of \mathbb{S}_n , each \mathbb{S}_n -isotypic components of type μ affords the structure of a GL_∞ -module. Let us denote by \mathfrak{c}_μ this \mathbb{S}_n -isotypic component. Its character is

$$\mathfrak{c}_\mu = \sum_{\lambda} c_{\lambda, \mu} s_\lambda,$$

⁴The rational case is the coprime special case of the more general rectangular $m \times n$ case.

which is simply the **coefficient** of $s_\mu(\mathbf{z})$ in $\mathcal{E}_n(\mathbf{q}; \mathbf{z})$. The character of $\mathcal{E}_n^{\langle k \rangle}$ is simply obtained as the evaluation:

$$\mathcal{E}_n(\mathbf{q}; \mathbf{z}) = \sum_{\mu \vdash n} \mathbf{c}_\mu(\mathbf{q}) s_\mu(\mathbf{z}), \quad (1.7)$$

in k parameters $\mathbf{q} = q_1, \dots, q_k$ (thus k is specified), and formal variables $\mathbf{z} = (z_i)_{i \in \mathbb{N}}$. In particular,

$$\mathcal{E}_n(q; \mathbf{z}) = \frac{h_n[\mathbf{z}/(1-q)]}{h_n[1/(1-q)]}, \quad \text{and} \quad \mathcal{E}_n(q, t; \mathbf{z}) = \nabla(e_n)(q, t; \mathbf{z}). \quad (1.8)$$

More aspects of this will be discussed below.

A crucial missing part in our previous work on the general case was an explicit description of the irreducible decomposition of \mathcal{E}_n (the values of the $c_{\lambda, \mu}$). We are currently going to describe large portions of this decomposition, characterized by a strikingly small set of data. This involves a precise and explicit link between various \mathbb{S}_n -isotypic components of the modules under study, established in part via Vertex Operators (see [32]). We also explain how this may be extended to all shapes μ having at most two parts larger than 1. A surprising corollary of Identity (2.8) is that we can reconstruct the alternating component of \mathcal{E}_n from the only knowledge of (hook-shape components) of the module $\mathcal{E}_n^{\langle k \rangle}$, with $k = \lfloor \frac{n-1}{2} \rfloor$ (rather than having to calculate up to $k = n - 1$).

Many seemingly independent aspects of the theory for $k = 2$ are nicely explained and tied together via general properties of the \mathcal{E}_n and their decomposition into irreducible components. As we will see most of these ties cannot be explained if one stays in the restricted context of $k = 2$; it is only by going to the general stable framework that the simplicity of the underlying structure is revealed. In particular, we establish a surprising connection with the ‘‘Delta operators’’ Δ_{e_k} (see [8]), which generalize the ∇ operator, shedding new light on an open conjecture of Haglund-Remmel-Wilson (see [17]). Indeed, one of our main conjecture states that

Conjecture 1 (Delta-via-Skewing). *For $0 \leq j \leq n - 1$, we have*

$$((e_j^\perp \otimes \text{Id}) \mathcal{E}_n)(q, t; \mathbf{z}) = \Delta'_{e_{n-1-j}} e_n(\mathbf{z}). \quad (1.9)$$

As we will see, this suggest how to construct modules corresponding to the $\Delta'_{e_k} e_n$. Moreover, there is a natural local version of this conjecture that arises in conjunction with the introduction of multivariate extensions of LLT-polynomials. This also establishes a connection with the ‘‘Superspace’’ of Bosons-Fermions.

For those that are either new to this subject, use different notation conventions, or work with other tools, we have taken care to give many explicit values; some appear in the text as such, and others are presented in the Appendix. Some background material may be found in [4].

2. EXPLICIT EXPRESSIONS

Even before describing the actual underlying modules, it seems best to start by giving a list of explicit values of \mathcal{E}_n 's, for small values of n . In our experience, this seems to be the best way to start understanding what this is all about. For $n \leq 5$, we have:

$$\begin{aligned}
\mathcal{E}_0 &= 1 \otimes 1, \\
\mathcal{E}_1 &= 1 \otimes s_1, \\
\mathcal{E}_2 &= 1 \otimes s_2 + s_1 \otimes s_{11}, \\
\mathcal{E}_3 &= 1 \otimes s_3 + (s_1 + s_2) \otimes s_{21} + (s_{11} + s_3) \otimes s_{111}, \\
\mathcal{E}_4 &= 1 \otimes s_4 + (s_1 + s_2 + s_3) \otimes s_{31} + (s_{21} + s_2 + s_4) \otimes s_{22} \\
&\quad + (s_{11} + s_{21} + s_{31} + s_3 + s_4 + s_5) \otimes s_{211} + (s_{111} + s_{31} + s_{41} + s_6) \otimes s_{1111}, \\
\mathcal{E}_5 &= 1 \otimes s_5 + (s_1 + s_2 + s_3 + s_4) \otimes s_{41} \\
&\quad + (s_{22} + s_{21} + s_{31} + s_{41} + s_2 + s_3 + s_4 + s_5 + s_6) \otimes s_{32} \\
&\quad + (s_{32} + s_{11} + s_{21} + 2s_{31} + s_{41} + s_{51} + s_3 + s_4 + 2s_5 + s_6 + s_7) \otimes s_{311} \\
&\quad + (s_{211} + s_{311} + s_{22} + s_{32} + s_{42} \\
&\quad\quad + s_{21} + s_{31} + 2s_{41} + 2s_{51} + s_{61} + s_4 + s_5 + s_6 + s_7 + s_8) \otimes s_{221} \\
&\quad + (s_{111} + s_{211} + s_{311} + s_{411} + s_{33} + s_{32} + s_{42} + s_{52} \\
&\quad\quad + s_{31} + 2s_{41} + 2s_{51} + 2s_{61} + s_6 + s_7 + s_{71} + s_8 + s_9) \otimes s_{2111} \\
&\quad + (s_{1111} + s_{311} + s_{411} + s_{511} \\
&\quad\quad + s_{43} + s_{42} + s_{62} + s_{61} + s_{71} + s_{81} + s_{(10)}) \otimes s_{11111}.
\end{aligned}$$

We may consider as a formal Schur \otimes Schur expansions of $\nabla(e_n)$ the restriction of the above expressions to the λ 's that have at most 2 parts; and then write more simply

$$\begin{aligned}
\mathcal{E}_n &= \nabla(e_n), \quad \text{for } 0 \leq n \leq 3, \\
\mathcal{E}_4 &= \nabla(e_4) + s_{111} \otimes s_{1111}, \\
\mathcal{E}_5 &= \nabla(e_5) + (s_{211} + s_{311}) \otimes s_{221} \\
&\quad + (s_{111} + s_{211} + s_{311} + s_{411}) \otimes s_{2111} \\
&\quad + (s_{1111} + s_{311} + s_{411} + s_{511}) \otimes s_{11111}.
\end{aligned}$$

In particular, this allows us to express \mathcal{E}_6 in a reasonably compact manner as:

$$\begin{aligned}
\mathcal{E}_6 = & \nabla(e_6) + (s_{221} + s_{411}) \otimes s_{33} \\
& + (s_{221} + 2s_{321} + s_{421} + s_{211} + 2s_{311} + 2s_{411} + 2s_{511} + s_{611}) \otimes s_{321} \\
& + (s_{331} + s_{321} + s_{421} + s_{521} + s_{111} + s_{211} + 2s_{311} + 2s_{411} \\
& \quad + 2s_{511} + s_{611} + s_{711}) \otimes s_{3111} \\
& + (s_{3111} + s_{331} + s_{221} + s_{321} + s_{421} + s_{521} + s_{311} + s_{411} \\
& \quad + 2s_{511} + s_{611} + s_{711}) \otimes s_{222} \\
& + (s_{2111} + s_{3111} + s_{4111} + s_{331} + s_{431} + s_{221} + 2s_{321} + 3s_{421} + 2s_{521} + s_{621} \\
& \quad + s_{211} + s_{311} + 3s_{411} + 3s_{511} + 4s_{611} + 2s_{711} + s_{811}) \otimes s_{2211} \\
& + (s_{1111} + s_{2111} + s_{3111} + s_{4111} + s_{5111} + s_{331} + 2s_{431} + s_{531} \\
& \quad + s_{321} + 2s_{421} + 2s_{521} + 2s_{621} + s_{721} + s_{311} + 2s_{411} \\
& \quad + 3s_{511} + 3s_{611} + 3s_{711} + 2s_{811} + s_{911}) \otimes s_{21111} \\
& + (s_{11111} + s_{3111} + s_{4111} + s_{5111} + s_{6111} + s_{441} + s_{431} \\
& \quad + s_{531} + s_{631} + s_{421} + s_{521} + s_{621} + s_{721} + s_{821} \\
& \quad + s_{611} + s_{711} + 2s_{811} + s_{911} + s_{(10,1,1)}) \otimes s_{111111}.
\end{aligned}$$

2.1. Numerical Specializations. Using the well-known evaluation

$$s_\mu(k) = s_\mu(\underbrace{1, 1, \dots, 1}_{k \text{ copies}}, 0, \dots) = \prod_{(i,j) \in \mu} \frac{k + (j - i)}{h(i, j)}, \quad (2.1)$$

where (i, j) runs over the set of cells of μ , and $h(i, j)$ stands for the associated **hook length**, we get a polynomial expression

$$\mathcal{E}_n(k; \mathbf{z}) := \mathcal{E}_n(\underbrace{1, 1, \dots, 1}_{k \text{ copies}}, 0, \dots; \mathbf{z}), \quad (2.2)$$

in the variable k . For instance,

$$\mathcal{E}_2(k; \mathbf{z}) = k s_{11}(\mathbf{z}) + s_2(\mathbf{z}),$$

and

$$\mathcal{E}_3(k; \mathbf{z}) = \left(\binom{k}{3} + 3 \binom{k}{2} + k \right) s_{111}(\mathbf{z}) + \left(\binom{k}{2} + 2k \right) s_{21}(\mathbf{z}) + s_3(\mathbf{z}).$$

Since it corresponds to the **alternating component** of \mathcal{E}_n , it is natural to denote by \mathcal{A}_n the coefficient of s_{1^n} in \mathcal{E}_n . In terms of the classical Hall scalar product on symmetric functions, it is then natural to set

$$\dim \mathcal{E}_n^{(k)} := \langle \mathcal{E}_n(k; \mathbf{z}), p_1(\mathbf{z})^n \rangle, \quad \text{and} \quad \dim \mathcal{A}_n^{(k)} := \langle \mathcal{E}_n(k; \mathbf{z}), e_n(\mathbf{z}) \rangle.$$

In view of earlier discussion, one sees that the symmetric function $\mathcal{E}_n(k; \mathbf{z})$ arises as the Frobenius characteristic of the \mathbb{S}_n -module $\mathcal{E}_n^{(k)}$. When $k = 1$, this module is the regular representation of \mathbb{S}_n , for which one has the classical formula

$$\mathcal{E}_n(1; \mathbf{z}) = e_1(\mathbf{z})^n = \sum_{\mu \vdash n} f^\mu s_\mu(\mathbf{z}), \quad (2.3)$$

with f^μ denoting the number of standard Young tableaux of shape μ , which may be calculate using the hook-length formula. For $k = 2$, we also get a “well-known” formula (see for instance [28]) here expressed both in compact form, using plethystic notation (see Appendix), and in extenso as:

$$\mathcal{E}_n(2; \mathbf{z}) = \frac{1}{n+1} e_n[(n+1)\mathbf{z}] = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} (n+1)^{\ell(\mu)-1} z_\mu^{-1} p_\mu(\mathbf{z}). \quad (2.4)$$

For $k = 3$ the following formula is conjectured to hold (see [9])

$$\mathcal{E}_n(3; \mathbf{z}) = \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} (n+1)^{\ell(\mu)-2} z_\mu^{-1} p_\mu(\mathbf{z}) \prod_{k \in \mu} \binom{2k}{k}. \quad (2.5)$$

No similar nice formula is known for larger values of k . Clearly, Formula (2.4) implies that

$$\dim \mathcal{E}_n^{(2)} = (n+1)^{n-1}, \quad \text{and} \quad \dim \mathcal{A}_n^{(2)} = \frac{1}{n+1} \binom{2n}{n}; \quad (2.6)$$

and Formula (2.5) implies

$$\dim \mathcal{E}_n^{(3)} = 2^n (n+1)^{n-1}, \quad \text{and} \quad \dim \mathcal{A}_n^{(3)} = \frac{2}{n(n+1)} \binom{4n+1}{n-1}. \quad (2.7)$$

Up to now, no formulas is known for $\dim \mathcal{E}_n^{(k)}$ or $\dim \mathcal{A}_n^{(k)}$, when $k \geq 4$ (however, see [10]). However, the stableness of (1.1) (discussed in the sequel) implies that the dimension of $\mathcal{E}_n^{(k)}$ (and $\mathcal{A}_n^{(k)}$) is polynomial in k , for each pair n . For instance, we have the following positive integer linear combinations of binomial coefficient polynomials

$$\begin{aligned} \dim \mathcal{E}_2^{(k)} &= k+1, \\ \dim \mathcal{A}_2^{(k)} &= k; \\ \dim \mathcal{E}_3^{(k)} &= \binom{k}{3} + 5 \binom{k}{2} + 5k + 1, \\ \dim \mathcal{A}_3^{(k)} &= \binom{k}{3} + 3 \binom{k}{2} + k; \\ \dim \mathcal{E}_4^{(k)} &= \binom{k}{6} + 12 \binom{k}{5} + 51 \binom{k}{4} + 96 \binom{k}{3} + 78 \binom{k}{2} + 23k + 1, \\ \dim \mathcal{A}_4^{(k)} &= \binom{k}{6} + 9 \binom{k}{5} + 25 \binom{k}{4} + 29 \binom{k}{3} + 12 \binom{k}{2} + k. \end{aligned}$$

It would be nice to have an explicit combinatorial understanding of these expressions in general.

2.2. Main results-conjectures. Considering the “scalar product” such that $\langle f \otimes s_\nu, s_\mu \rangle = f$, so that $\langle \mathcal{E}_n, s_\mu \rangle$ is the coefficient of s_μ in \mathcal{E}_n , we may express our first main “fact” as follows:

Conjecture 2 (Hook-Components). *For all n and all $0 \leq k \leq n-1$, if μ is the hook shape $(k+1, 1^{n-k-1})$, then we have the identity*

$$e_k^\perp \mathcal{A}_n = \langle \mathcal{E}_n, s_\mu \rangle. \quad (2.8)$$

In particular, $e_{n-1}^\perp \mathcal{A}_n = 1$.

One of the interesting implication of (2.8), together with Conjecture.3 below, is that we can reconstruct \mathcal{A}_n from (very) partial knowledge of the values of the $\langle \mathcal{E}_n, s_\mu \rangle$. To see how this goes, let us first state the following conjecture, defining the **length** $\ell(f)$ of a symmetric function f , to be the maximum number of parts $\ell(\lambda)$ in a partition λ that index a Schur function s_λ occurring with non-zero coefficient a_λ in its Schur expansion $f = \sum_\lambda a_\lambda f_\lambda$. In formula:

$$\ell(f) = \max_{a_\lambda \neq 0} \ell(\lambda).$$

Conjecture 3 (Coefficient-Length). *For all partition μ of n , we have $\ell(\langle \mathcal{E}_n, s_\mu \rangle) = n - \mu_1$. In particular, $(n-k-1)$ is the length of the coefficient of s_μ , for the hook-shape $\mu = (k+1, 1^{n-k-1})$.*

For $k = n - 1$, this is compatible with Theorem 2 which implies that

$$\mathcal{A}_n = s_{1^{n-1}} + (\text{lower length terms}),$$

so that \mathcal{A}_n is indeed of length $n - 1$. As another example, the length of $\langle \mathcal{E}_n, s_{(n-1,1)} \rangle$ is conjectured to be equal to 1, so that we get

$$\langle \mathcal{E}_n, s_{(n-1,1)} \rangle = \langle \nabla(e_n), s_{(n-1,1)} \rangle = s_1 + s_2 + \dots + s_{n-1}, \quad (2.9)$$

since the second equality is well known.

2.3. \mathcal{A}_n -reconstruction. Let us illustrate how, assuming the Coefficient-Length Conjecture, we may reconstruct \mathcal{A}_n . We already know that in general $\mathcal{A}_n = s_{1^{n-1}} + (\text{lower length terms})$, so that $e_{n-2}^\perp \mathcal{A}_n = s_1 + (\text{other length 1 terms})$. In fact, from (2.9), we also get for all n that

$$e_{n-2}^\perp \mathcal{A}_n = s_1 + s_2 + \dots + s_{n-1},$$

which forces

$$\mathcal{A}_n = s_{1^{n-1}} + \sum_{k=3}^n s_{(k,1^{n-3})} + (\text{terms of length } < n - 2). \quad (2.10)$$

Likewise, all terms of length $n - 3$ of \mathcal{A}_n are imposed by the identity

$$e_{n-3}^\perp \mathcal{A}_n = \langle \mathcal{E}_n, s_{n-2,1,1} \rangle = \langle \nabla(e_n), s_{n-2,1,1} \rangle, \quad (2.11)$$

which results from the assumption that $\langle \mathcal{E}_n, s_{n-2,1,1} \rangle$ is of length 2, hence its value is entirely determined by $\nabla(e_n)$. For instance, with $n = 6$, this gives

$$\begin{aligned} \mathcal{A}_6 = & s_{111111} + s_{31111} + s_{41111} + s_{51111} + s_{61111} \\ & + s_{441} + s_{431} + s_{531} + s_{631} + s_{421} + s_{521} + s_{621} + s_{721} + s_{821} \\ & + s_{611} + s_{711} + 2s_{811} + s_{911} + s_{10.11} + (\text{terms of length } \leq 2). \end{aligned}$$

The remaining missing terms are thus readily calculated since they correspond exactly to the Schur expansion of $\langle \nabla(e_6), e_6 \rangle$, which we give below for completeness sake.

$$\begin{aligned} \langle \nabla(e_6), e_6 \rangle = & s_{44} + s_{64} + s_{74} + s_{63} + s_{73} + s_{83} + s_{93} + s_{72} + s_{82} + s_{92} + s_{10.2} + s_{11.2} \\ & + s_{10.1} + s_{11.1} + s_{12.1} + s_{13.1} + s_{15}. \end{aligned}$$

Once again, let us underline that for all n , the value of $\nabla(e_n)$ fixes all the components of length at most 2 in \mathcal{E}_n .

2.4. Hook-components reconstruction. Now, from this explicit knowledge of \mathcal{A}_n , we may calculate all the hook-indexed components of \mathcal{E}_n . In particular, we may check that we indeed get back the expression already mentioned for $\langle \mathcal{E}_6, s_{21111} \rangle$, namely

$$\begin{aligned} \langle \mathcal{E}_6, s_{21111} \rangle &= e_1^\perp \mathcal{A}_n \\ &= \langle \nabla(e_6), s_{21111} \rangle + (s_{1111} + s_{2111} + s_{3111} + s_{4111} + s_{5111} + s_{331} + 2s_{431} + s_{531} \\ &\quad + s_{321} + 2s_{421} + 2s_{521} + 2s_{621} + s_{721} + s_{311} + 2s_{411} \\ &\quad + 3s_{511} + 3s_{611} + 3s_{711} + 2s_{811} + s_{911}); \end{aligned}$$

as well as that for $\langle \mathcal{E}_6, s_{3111} \rangle$:

$$\begin{aligned} \langle \mathcal{E}_6, s_{3111} \rangle &= e_2^\perp \mathcal{A}_n \\ &= \langle \nabla(e_6), s_{3111} \rangle + (s_{331} + s_{321} + s_{421} + s_{521} + s_{111} + s_{211} + 2s_{311} + 2s_{411} \\ &\quad + 2s_{511} + s_{611} + s_{711}). \end{aligned}$$

2.5. Partial reconstruction of other components. Similarly, using both the Component-Length Conjecture and the Delta-via-Skewing Conjecture (see [1](#)), we may partially reconstruct other coefficients of \mathcal{E}_n , considering that the expansion of $\Delta'_{e_k}(e_n)$ is known for all k and n . Observe that the Component-Length Conjecture directly implies that, for all n ,

$$\langle \mathcal{E}_n, s_{n-2,2} \rangle = \langle \nabla(e_n), s_{n-2,2} \rangle \quad (2.12)$$

so that we already have the coefficient of $s_{n-2,2}$ fully characterized, on top of those for all hook-shapes. Since the Delta-via-Skewing Conjecture states that the length at most 2 components of $\langle e_k^\perp \mathcal{E}_n, s_\mu \rangle$ coincide with those of $\langle \Delta'_{e_{n-1-k}} e_n, s_\mu \rangle$ for all μ , it may be used to infer components of the corresponding coefficients. We may also deduce from [Conjecture.1](#) part of [Conjecture.3](#). For instance, since $\Delta'_{e_1} e_n = \Delta_{e_1} e_n - 1 \otimes e_n$ and we have⁵ (see [[17](#), Prop. 6.1])

$$\Delta_{e_1}(e_n) = \sum_{k=1}^n s_{k-1} \otimes e_{n-k} e_k, \quad (2.13)$$

we deduce that $\langle \Delta'_{e_1} e_n, s_\mu \rangle = 0$ for all partition μ having first part larger than 2. Hence, [Conjecture.1](#) implies that $\langle e_{n-3}^\perp \mathcal{E}_n, s_\mu \rangle = 0$ when $\mu_1 > 2$, implying that $\langle \mathcal{E}_n, s_\mu \rangle = 0$ has length at most 2 in those cases. However, for μ such that $\mu_1 = 2$, [Formula \(2.13\)](#) implies that $\langle e_{n-2}^\perp \mathcal{E}_n, s_{(2^k, 1^{n-2k})} \rangle$ does not vanish and is of length 1. Thus we conclude that

Lemma 2.1. *Conjecture.1 implies Conjecture.3, for any partition μ such that $\mu_1 = 2$.*

Moreover, using [\(2.13\)](#), [Conjecture.1](#) states that

$$\langle e_{n-2}^\perp \mathcal{E}_n, s_{(2^k, 1^{n-2k})} \rangle = \sum_{i=k-1}^{n-k-1} s_i.$$

Since we already know that $\langle e_{n-2}^\perp \mathcal{E}_n, s_{(2^k, 1^{n-2k})} \rangle = 0$ if $k \geq 1$, the above identity forces

$$\langle \mathcal{E}_n, s_{(2^k, 1^{n-2k})} \rangle = \sum_{i=k-1}^{n-k-1} s_{i+1, 1^{n-3}} + (\text{terms of length } < n-2). \quad (2.14)$$

⁵Since it underlines Schur positivity in the parameters q and t , this is a “slightly” stronger statement than that of [[17](#)], but it is equivalent.

2.6. An intriguing Catalan property. An experimental observation is that at $k = -2$ our earlier polynomial $\dim(\mathcal{E}_n^{(k)})$ appear to evaluate to the signed Catalan numbers $(-1)^{n-1} \text{Cat}_{n-1}$. Moreover, this intriguing signed Catalan property seems to afford the refinement:

$$\mathcal{E}_n(-2; \mathbf{z}) = (-1)^{n-1} \sum_{\mu \vdash n} \pi(\mu) \text{Cat}_{\ell(\mu)-1} f_\mu(\mathbf{z}), \quad (2.15)$$

where $\pi(\mu)$ is the product of the parts of μ , and f_μ denotes the **forgotten** symmetric function. We observe that this is the specialization at $q = 1$ of the formula

$$(-q)^{n-1} \mathcal{E}_n[-q - 1/q; \mathbf{z}] = \sum_{\mu \vdash n} \prod_{k \in \mu} ([k]_{q^2}) C_{\ell(\mu)}(-q) f_\mu(\mathbf{z}) \quad (2.16)$$

where

$$C_n(q) := \sum_{k=0}^{2n-1} \binom{n}{\lfloor (k-1)/2 \rfloor} \binom{n}{\lfloor k/2 \rfloor} q^{k-1};$$

since we have $C_n(-1) = \text{Cat}_{n-1}$.

3. SPACES OF MULTIVARIATE DIAGONAL HARMONIC POLYNOMIALS

To make this presentation more self contained, let us recall a few basic definitions⁶. Let \mathbf{X} be a $k \times n$ matrix of variables, so that we may better underline the $\text{GL}_k \times \mathbb{S}_n$ action on polynomials in the variables \mathbf{X} :

$$\mathbf{X} = (v_{ij})_{\substack{1 \leq i \leq k \\ 1 \leq j \leq n}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ \vdots & \vdots & \ddots & \vdots \\ u_1 & u_2 & \cdots & u_n \end{pmatrix}.$$

It will be handy to denote by \mathbf{x} (respectively, \mathbf{y} and \mathbf{u}) the set of variables in the first row of \mathbf{X} (respectively, second and “last”). We will consider $\text{GL}_k \times \mathbb{S}_n$ -submodules of the ring $\mathcal{R}_n^{(k)} := \mathbb{Q}[\mathbf{X}]$ of polynomials $F(\mathbf{X})$ in the variables in \mathbf{X} , equipped with the action of the group $\text{GL}_k \times \mathbb{S}_n$ defined by

$$F(\mathbf{X}) \mapsto F(\tau \cdot \mathbf{X} \cdot \sigma; \boldsymbol{\theta} \cdot \sigma), \quad \text{for } (\tau, \sigma) \in \text{GL}_k \times \mathbb{S}_n,$$

with elements of \mathbb{S}_n considered as permutation matrices in GL_n . It is sometimes useful to consider each row $\mathbf{u} = (u_1, \dots, u_n)$ of \mathbf{X} as a set of n variables, permuted by the \mathbb{S}_n -action. The actions of GL_k and \mathbb{S}_n on $\mathcal{R}_n^{(k)}$ commute, hence we may readily decompose the modules considered into irreducibles for both actions. A polynomial $F(\mathbf{X})$ in $\mathcal{R}_n^{(k)}$ is said to be (multi-)homogeneous of degree $\mathbf{d} \in \mathbb{N}^k$, if

$$F(\mathbf{q} \cdot \mathbf{X}) = \mathbf{q}^{\mathbf{d}} F(\mathbf{X}),$$

with \mathbf{q} standing for the diagonal matrix $\text{diag}(q_1, \dots, q_k)$. We often use “vector-like notations”, so that $\mathbf{q}^{\mathbf{d}}$ stands for $q_1^{d_1} \cdots q_k^{d_k}$, when $\mathbf{d} = (d_1, \dots, d_k)$. In the context of polynomial representations of GL_k , the vectors \mathbf{d} are **weights**.

⁶General background material may be found in the Appendix.

The $(\mathrm{GL}_k \times \mathbb{S}_n)$ -modules that we are going to consider are all graded submodules (or graded quotients of such) of $\mathcal{R}_n^{(k)}$, and hence they correspond to polynomial representations of GL_k . Thus the relevant GL_k -characters take the form

$$\sum_{F \in \mathcal{B}} \mathbf{q}^{\deg(F)}, \quad \text{with} \quad \deg(F) := (\deg_{\mathbf{x}}(F), \deg_{\mathbf{y}}(F), \dots, \deg_{\mathbf{u}}(F)),$$

for some (homogeneous) basis \mathcal{B} of the GL_k -module considered. Classical representation theory gives that this sum is a symmetric function of the variables $\mathbf{q} = (q_1, \dots, q_k)$, which expands as a positive integer linear combination of Schur functions $s_\lambda(\mathbf{q}) = s_\lambda(q_1, \dots, q_k)$, for some partitions λ . As usual, positive integral linear combination of Schur functions are said to be **Schur positive** (see Appendix 6 for more on symmetric functions). It is also practical to encode characters of representations of \mathbb{S}_n as symmetric functions⁷ via their Frobenius transform, with irreducible \mathbb{S}_n -modules corresponding to Schur functions $s_\mu(\mathbf{z})$, for μ partitions of n . The description of the irreducible decompositions of a $(\mathrm{GL}_k \times \mathbb{S}_n)$ -submodule \mathcal{W} of $\mathcal{R}_n^{(k)}$, may thus be presented as:

$$\mathcal{W}(\mathbf{q}; \mathbf{z}) = \sum_{\mu \vdash n} \sum_{\lambda} a_{\lambda\mu} s_\lambda(\mathbf{q}) s_\mu(\mathbf{z}),$$

with the $a_{\lambda\mu}$ multiplicities of $(\mathrm{GL}_k \times \mathbb{S}_n)$ -irreducibles. Since the number of variables in the set \mathbf{q} is equal to k , this information needs not be underlined otherwise. As discussed in the introduction, we are interested in stable limits of such characters $\mathcal{W} = \lim_{k \rightarrow \infty} \mathcal{W}_k$, then writing

$$\mathcal{W} = \sum_{\mu \vdash n} \sum_{\lambda} a_{\lambda\mu} s_\lambda \otimes s_\mu,$$

with the tensor product serving to distinguish between Schur functions that are characters of GL_∞ (those on left hand-side), and the Schur function encoding \mathbb{S}_n -irreducibles (those on right hand-side).

3.1. Diagonal harmonic polynomials. Starting with the classical $n \times n$ **Vandermonde determinant** in the variables occurring in the first row of \mathbf{X} :

$$\begin{aligned} \mathbf{V}_n &:= \det(x_i^{j-1})_{1 \leq i, j \leq n}, \\ &= \prod_{i < j} x_i - x_j, \end{aligned} \tag{3.1}$$

we consider the smallest submodule $\mathcal{E}_n^{(k)}$ of $\mathcal{R}_n^{(k)}$ which contains \mathbf{V}_n , and is closed under:

- (1) partial derivatives with respect to any variables in \mathbf{X} ;
- (2) **higher polarization** operators:

$$\sum_{i=1}^n u_i^j \frac{\partial}{\partial v_i},$$

for any pair of rows $(u_i)_i$ and $(v_i)_i$ of \mathbf{X} , and any $j \geq 1$.

⁷In a denumerable set of “abstract” variables $\mathbf{z} = z_1, z_2, z_3, \dots$

It is easy to see that $\mathcal{E}_n^{(k)}$ is a \mathbb{S}_n -submodule of $\mathcal{R}_n^{(k)}$; and closure under polarization insures that it is also a submodule for GL_k . In fact, closure under all operators $\sum_{i=1}^n u_i \frac{\partial}{\partial v_i}$, i.e. with $j = 1$, is sufficient.

It is clear that we have a filtration

$$\mathcal{E}_n^{(0)} \subset \mathcal{E}_n^{(1)} \subset \mathcal{E}_n^{(2)} \subset \dots \mathcal{E}_n^{(k)} \subset \dots$$

compatible with the action \mathbb{S}_n ; and with the restriction from GL_k to GL_{k-1} , since this corresponds to the restriction to the polynomials involving only the first $k-1$ rows of \mathbf{X} . We denote by \mathcal{E}_n the corresponding inductive limit, and it has been shown in [5] that one reaches stability at $k = n - 1$. As mentioned earlier, the corresponding ‘‘generic’’ character is denoted by

$$\mathcal{E}_n = \sum_{\mu \vdash n} \sum_{\lambda} c_{\lambda, \mu} s_{\lambda} \otimes s_{\mu}. \quad (3.2)$$

4. STRUCTURE PROPERTIES OF \mathcal{E}_n

4.1. **Length components.** Define the **degree** of

$$\sum_{\lambda, \mu} c_{\lambda, \mu} s_{\lambda} \otimes s_{\mu}$$

to be the maximum of the values $|\lambda|$, for which $c_{\lambda, \mu} \neq 0$. The **length d component** of \mathcal{E}_n is set to be

$$\mathcal{E}_n^{(d)} := \sum_{\mu \vdash n} \mathbf{c}_{\mu}^{(d)} \otimes s_{\mu}, \quad \text{with} \quad \mathbf{c}_{\mu}^{(d)} = \sum_{\ell(\lambda)=d} c_{\lambda, \mu} s_{\lambda}. \quad (4.1)$$

We clearly have

$$\mathcal{E}_n := \mathcal{E}_n^{(0)} + \mathcal{E}_n^{(1)} + \dots \mathcal{E}_n^{(l)}, \quad (4.2)$$

where $l = \ell(\mathcal{E}_n)$ is the maximal length occurring in terms of \mathcal{E}_n . With this notation, Conjecture.3 states that $\ell(\mathbf{c}_{\mu}) = |\mu| - \mu_1$, and it may be shown that

$$\deg(\mathbf{c}_{\mu}^{(j)}) = \binom{n}{2} - \binom{j}{2} - \sum_{i \in \mu} \binom{i}{2}. \quad (4.3)$$

Let $\mathcal{E}_n^{(k)}$ (similarly for $\mathbf{c}_{\mu}^{(k)}$) stand for the restriction of \mathcal{E}_n to its components of length at most k . In other terms,

$$\mathcal{E}_n^{(k)} := \mathcal{E}_n^{(0)} + \mathcal{E}_n^{(1)} + \dots \mathcal{E}_n^{(k)}. \quad (4.4)$$

For example,

$$\begin{aligned} \mathcal{E}_4^{(0)} &= 1 \otimes s_4, \\ \mathcal{E}_4^{(1)} &= (s_1 + s_2 + s_3) \otimes s_{31} + (s_2 + s_4) \otimes s_{22} + (s_3 + s_4 + s_5) \otimes s_{211} + s_6 \otimes s_{1111}, \\ \mathcal{E}_4^{(2)} &= s_{21} \otimes s_{22} + (s_{11} + s_{21} + s_{31}) \otimes s_{211} + (s_{31} + s_{41}) \otimes s_{1111}, \\ \mathcal{E}_4^{(3)} &= s_{111} \otimes s_{1111}. \end{aligned}$$

Simple general values for length components are:

$$\mathcal{E}_n^{(0)} = 1 \otimes s_n, \quad \text{and} \quad \mathcal{E}_n^{(n-1)} = e_{n-1} \otimes e_n. \quad (4.5)$$

The following **reduced-length components** can be efficiently used to reconstruct (part of) \mathcal{E}_n . We set

$$\varepsilon_n^{(k)} := (e_k^\perp \otimes \text{Id}) \mathcal{E}_n^{(k)}, \quad (\text{and} \quad \alpha_n^{(k)} := e_k^\perp \mathcal{A}_n^{(k)}). \quad (4.6)$$

For example,

$$\begin{aligned} \varepsilon_4^{(0)} &= 1 \otimes s_4, \\ \varepsilon_4^{(1)} &= (1 + s_1 + s_2) \otimes s_{31} + (s_1 + s_3) \otimes s_{22} + (s_2 + s_3 + s_4) \otimes s_{211} + s_5 \otimes s_{1111}, \\ \varepsilon_4^{(2)} &= (1 + s_1 + s_2) \otimes s_{211} + s_1 \otimes s_{22} + (s_2 + s_3) \otimes s_{1111}, \\ \varepsilon_4^{(3)} &= 1 \otimes s_{1111}. \end{aligned}$$

4.2. Description of hook components in terms of \mathcal{A}_n . From (2.8) we may calculate all the hook components $\mathbf{c}_\mu = \langle \mathcal{E}_n, s_\mu \rangle$ directly from the alternant component $\mathcal{A}_n = \langle \mathcal{E}_n, s_{1^n} \rangle$, both considered as GL_∞ -character of isotypic components of \mathcal{E}_n . We will see that this implies that we can reconstruct \mathcal{A}_n from much less information than is apparently needed at *prima facie*. From now on, let us use **Frobenius's notation** for hook shape partitions, writing $(a | b)$ for the partition $(a + 1, 1^b)$ of $n = a + b + 1$ (see Figure 1). It is often said that a stands for the **arm** of the hook, while b stands for its **leg**. We use the Cartesian (aka French) convention to draw diagrams, so that the leg goes up.

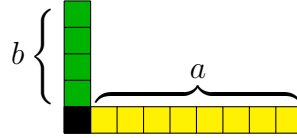


FIGURE 1. The hook shape $(a | b)$.

4.3. Conjectured formula for the hook part multiplicities in hook components.

Our calculations suggest that there is a simple elegant expression for the multiplicity $\langle s_\lambda, \mathbf{c}_\mu \rangle$ of $s_\lambda \otimes s_\mu$ in \mathcal{E}_n , when both $\lambda = (i, 1^j)$ and $\mu = (n - k, 1^k)$ are hook shapes. Clearly we may encode bijectively a hook-shape $(i, 1^j)$ as a monomial $q^i u^j$. Regarding this encoding, it is interesting to observe that one has the plethystic evaluation⁸

$$s_{(i,1^j)}[q - \varepsilon u] = (q + u)q^{i-1}u^j. \quad (4.7)$$

For a given μ , one may thus package all the multiplicities $\langle s_{(i,1^j)}, \mathbf{c}_\mu \rangle$, for the various hooks $(i, 1^j)$, as a generating polynomial given simply by the expression

$$\frac{q}{q + u} \mathbf{c}_\mu[q - \varepsilon u] := \sum_{(i,1^j)} \langle s_{(i,1^j)}, \mathbf{c}_\mu \rangle q^i u^j. \quad (4.8)$$

This is readily calculated if we have the following.

Conjecture 4. For all $0 \leq k \leq n - 1$, we have

$$\mathbf{c}_{(n-k,1^k)}[q - \varepsilon u] = \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q (q + u)(q^2 + u) \cdots (q^k + u), \quad (4.9)$$

⁸Here ε is such that $p_k(\varepsilon) = (-1)^k$.

using the Gaussian q -binomial notation.

For example, the size collected restriction to hook shapes of $\langle \mathcal{E}_5, s_{2111} \rangle$ is

$$\begin{aligned} \mathbf{c}_{2111}|_{\text{hooks}} &= s_{111} + (s_{31} + s_{211}) + (2s_{41} + s_{311}) + (s_6 + 2s_{51} + s_{411}) \\ &\quad + (s_7 + 2s_{61}) + (s_8 + s_{71}) + s_9, \end{aligned}$$

and the corresponding polynomial is

$$\begin{aligned} \frac{q}{q+u} \mathbf{c}_\mu[q - \varepsilon u] &= \frac{q}{q+u} \left[\begin{matrix} 4 \\ 3 \end{matrix} \right]_q (q+u)(q^2+u)(q^3+u) \\ &= q(q+1)(q^2+1)(q^2+u)(q^3+u) \\ &= qu^2 + (q^3u + q^2u^2) + (2q^4u + q^3u^2) + (q^6 + 2q^5u + q^4u^2) \\ &\quad + (q^7 + 2q^6u) + (q^8 + q^7u) + q^9. \end{aligned}$$

When $\mu = 1^n$, Formula (4.9) follows from (2.8) (see [30]). In other words, we have

Proposition 1. *For all n , we have*

$$\mathbf{c}_{1^n}[q - \varepsilon u] = (q+u)(q^2+u) \cdots (q^{n-1}+u). \quad (4.10)$$

As another indication of the appropriateness of formula (4.9), we may directly check that its specialization at $u = 0$ does indeed correspond to the known expression for the hook components of $\mathcal{E}_n^{(1)}$. Indeed, the expression

$$\begin{aligned} \sum_{k=0}^{n-1} \mathbf{c}_{(n-k, 1^k)}[q] z^k &= \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \left[\begin{matrix} n-1 \\ k \end{matrix} \right]_q z^k, \\ &= \prod_{i=0}^{n-1} (1 + q^i z), \end{aligned}$$

may easily be seen to follow from formula (6.11), since $\mathcal{E}_n(q; \mathbf{z}) = H_n(q; \mathbf{z})$.

4.4. Other components. The GL_∞ -character of some of the other \mathbb{S}_n -isotypic components may as well be calculated from \mathcal{A}_n . For instance, this is the case when μ has at most two parts larger than 1. Indeed, for $\mu = ab1^c$, with a and b larger than 1, we have the recursive formula

$$\mathbf{c}_\mu = s_\alpha^\perp \mathcal{A}_n - \mathbf{c}_\beta, \quad (4.11)$$

where $\alpha := (a-1, b-1)' = 2^{b-1}1^{a-b}$ and $\beta := (a-1, b-1, 1^{c+1})$. Observe that β is a partition of $n-1$ which has the same format as μ , thus we may apply the formula recursively. For n up to 5, together with the hook cases considered previously, this covers all cases. For $n = 6$ the only exception is the partition $\mu = 222$. Observe also that, when $b = 2$, we may use formula (2.8) to reformulate (4.11) as

$$\mathbf{c}_\mu = s_\alpha^\perp \mathcal{A}_n - e_{a-2}^\perp \mathcal{A}_{n-1}, \quad (4.12)$$

since β is then a hook.

4.5. **Reconstruction of the alternant character.** We now further discuss the **length decomposition**

$$\mathcal{A}_n = \mathcal{A}_n^{(1)} + \mathcal{A}_n^{(2)} + \dots + \mathcal{A}_n^{(n-1)},$$

and the corresponding reduced-length components $\alpha_n^{(j)} := e_j^\perp \mathcal{A}_n^{(j)}$ of \mathcal{A}_n . We can clearly get $\mathcal{A}_n^{(j)}$ back from $\alpha_n^{(j)}$, since $\mathcal{A}_n^{(j)} = \uparrow_{1^j} \alpha_n^{(j)}$. Hence, we can reconstruct \mathcal{A}_n from the knowledge of its reduced-length components $\alpha_n^{(j)}$. Clearly $e_j^\perp \mathcal{A}_n^{(i)} = 0$ whenever $i < j$. In view of previous remarks concerning the length decomposition of the \mathbf{c}_μ , we have

$$l = \ell(\alpha_n^{(j)}) = \min(j, n-1-j). \quad (4.13)$$

In fact, the length-decomposition of $\alpha_n^{(j)}$ itself takes the form

$$\alpha_n^{(j)} = \alpha_n^{(0,j)} + \alpha_n^{(1,j)} + \dots + \alpha_n^{(l,j)},$$

and the degrees of these length components of the $\alpha_n^{(j)}$ are

$$\deg(\alpha_n^{(i,j)}) = \binom{n}{2} - \binom{j+1}{2} - \binom{i}{2}.$$

Using Conjecture 2.8, we get the following formulas for some of the reduced-length components of \mathcal{A}_n

$$\begin{aligned} 1) \alpha_n^{(1)} &= s_{\binom{n}{2}-1}, & 2) \alpha_n^{(2)} &= e_2^\perp \langle \nabla(e_n), e_n \rangle, \\ 3) \alpha_n^{(n-3)} &= \langle \nabla(e_n), s_{(n-2,1,1)} \rangle - e_1 \alpha_n^{(n-2)} - s_{11}, & 4) \alpha_n^{(n-2)} &= \sum_{k=2}^{n-1} s_k, \\ 5) \alpha_n^{(n-1)} &= 1; \end{aligned} \quad (4.14)$$

and, using (2.8) and (4.13), the recurrence

$$\alpha_n^{(j)} = \mathbf{c}_{\binom{d}{j|b}} - \sum_{i=j+1}^{n-1} e_j^\perp (\uparrow_{1^i} \alpha_n^{(i)}), \quad \text{where } b = n-1-j, \quad \text{and } d := \min(j, b). \quad (4.15)$$

Observe that (4.13) on $\alpha_n^{(i)}$ implies that

$$e_j^\perp (\uparrow_{1^i} \alpha_n^{(i)}) = e_{i-j} \alpha_n^{(i)}, \quad \text{when } j \leq i.$$

Hence, since $e_j^\perp \mathcal{A}_n^{(i)} = 0$ for $j > i$, we have in fact

$$\alpha_n^{(j)} = \mathbf{c}_{\binom{j}{j|b}} - \sum_{i=b+1}^{n-1} e_{i-b} \alpha_n^{(i)}, \quad (4.16)$$

whenever $j < (n-1)/2$. Observe that both (4.15) and (4.16) reduce the calculation of $\alpha_n^{(j)}$ to expressions of length at most $(n-1)/2$. This is illustrated in the following values of $\alpha_6^{(j)}$, all of length at most 2:

$$\begin{aligned} \alpha_6^{(5)} &= 1, \\ \alpha_6^{(4)} &= s_2 + s_3 + s_4 + s_5, \\ \alpha_6^{(3)} &= s_{33} + (s_{32} + s_{42} + s_{52}) + (s_{31} + s_{41} + s_{51} + s_{61} + s_{71}) \\ &\quad + (s_5 + s_6 + 2s_7 + s_8 + s_9), \end{aligned}$$

$$\alpha_6^{(2)} = (s_{33} + s_{53} + s_{63}) + (s_{52} + s_{62} + s_{72} + s_{82}) + (s_{61} + s_{71} + s_{81} + s_{91} + s_{(10,1)}) \\ + (s_9 + s_{(10)} + s_{(11)} + s_{(12)}),$$

$$\alpha_6^{(1)} = s_{(14)};$$

and we recover from these all components of \mathcal{A}_6 (of lengths going up to 5)

$$\begin{aligned} \mathcal{A}_6 = & s_{11111} + (s_{3111} + s_{4111} + s_{5111} + s_{6111}) \\ & + (s_{441} + s_{431} + s_{531} + s_{631} + s_{421} + s_{521} + s_{621} + s_{721} + s_{821} \\ & + s_{611} + s_{711} + 2s_{811} + s_{911} + s_{(10,1,1)}) \\ & + (s_{44} + s_{64} + s_{74} + s_{63} + s_{73} + s_{83} + s_{93} + s_{72} + s_{82} \\ & + s_{92} + s_{(10,2)} + s_{(11,2)} + s_{(10,1)} + s_{(11,1)} + s_{(12,1)} + s_{(13,1)}) + s_{(15)} \end{aligned} \quad (4.17)$$

To achieve a similar reconstruction for $n = 7$, the only extra information needed reduces to the knowledge of the length 3 component of the hook shape $(3|3)$. The required value has been explicitly computed (by N. Thiéry, using interesting computer algebra techniques) to be equal to

$$\begin{aligned} \mathbf{c}_{(3|3)} = & \langle \nabla(e_7, s_{4111}) \rangle + s_{332} + s_{522} \\ & + (2s_{331} + 2s_{431} + 2s_{531} + s_{631}) \\ & + (s_{321} + 2s_{421} + 3s_{521} + 2s_{621} + 2s_{721} + s_{821}) \\ & + (s_{111} + s_{211} + 2s_{311} + 3s_{411} + 3s_{511} + 3s_{611} + 3s_{711} + 2s_{811} + s_{911} + s_{(10,1,1)}). \end{aligned}$$

We may then calculate recursively the only “missing” reduced-length component of \mathcal{A}_6 (beside those calculated using formulas (4.14)), which is

$$\begin{aligned} \alpha_7^{(3)} = & (s_{332} + s_{522} + s_{331} + s_{431} + s_{531} + s_{631} \\ & + s_{421} + s_{521} + s_{621} + s_{721} + s_{821} + s_{411} + s_{611} + s_{711} + s_{811} + s_{(10,1,1)}) \\ & + (s_{55} + s_{54} + s_{64} + s_{74} + s_{33} + s_{43} + 2s_{53} + 3s_{63} + 3s_{73} + 2s_{83} + s_{93} \\ & + s_{52} + 2s_{62} + 3s_{72} + 3s_{82} + 3s_{92} + 2s_{(10,2)} + s_{(11,2)} \\ & + s_{61} + 2s_{71} + 2s_{81} + 3s_{91} + 3s_{(10,1)} + 2s_{(11,1)} + 2s_{(12,1)} + s_{(13,1)}) \\ & + (s_9 + s_{(10)} + 2s_{(11)} + 2s_{(12)} + 2s_{(13)} + s_{(14)} + s_{(15)}). \end{aligned}$$

4.6. Alternants of \mathcal{E}_7 . From this, all explicit values of length components of \mathcal{A}_7 may be obtained, giving

$$\begin{aligned} \mathcal{A}_7^{(1)} &= s_{(21)}; \\ \mathcal{A}_7^{(2)} &= s_{77} + (s_{76} + s_{86} + s_{96}) + (s_{75} + s_{85} + s_{95} + s_{(10,5)} + s_{(11,5)}) \\ &+ (s_{74} + s_{84} + 2s_{94} + 2s_{(10,4)} + 2s_{(11,4)} + s_{(12,4)} + s_{(13,4)}) \\ &+ (s_{93} + s_{(10,3)} + 2s_{(11,3)} + 2s_{(12,3)} + 2s_{(13,3)} + s_{(14,3)} + s_{(15,3)}) \\ &+ (s_{(11,2)} + s_{(12,2)} + 2s_{(13,2)} + s_{(14,2)} + 2s_{(15,2)} + s_{(16,2)} + s_{(17,2)}) \\ &+ (s_{(15,1)} + s_{(16,1)} + s_{(17,1)} + s_{(18,1)} + s_{(19,1)}); \end{aligned}$$

$$\begin{aligned}
\mathcal{A}_7^{(3)} &= s_{443} + s_{633} + (s_{442} + s_{542} + s_{642} + s_{742}) \\
&\quad + (s_{532} + s_{632} + s_{732} + s_{832} + s_{932}) \\
&\quad + (s_{522} + s_{722} + s_{822} + s_{922} + s_{(11,2,2)}) \\
&\quad + s_{661} + (s_{651} + s_{751} + s_{851}) \\
&\quad + (s_{441} + s_{541} + 2s_{641} + 3s_{741} + 3s_{841} + 2s_{941} + s_{(10,4,1)}) \\
&\quad + (s_{631} + 2s_{731} + 3s_{831} + 3s_{931} + 3s_{(10,3,1)} + 2s_{(11,3,1)} + s_{(12,3,1)}) \\
&\quad + (s_{721} + 2s_{821} + 2s_{921} + 3s_{(10,2,1)} + 3s_{(11,2,1)} \\
&\quad\quad + 2s_{(12,2,1)} + 2s_{(13,2,1)} + s_{(14,2,1)}) \\
&\quad + (s_{(10,1,1)} + s_{(11,1,1)} + 2s_{(12,1,1)} + 2s_{(13,1,1)} \\
&\quad\quad + 2s_{(14,1,1)} + s_{(15,1,1)} + s_{(16,1,1)}); \\
\mathcal{A}_7^{(4)} &= (s_{4411} + s_{5411} + s_{6411}) + (s_{4311} + s_{5311} + 2s_{6311} + s_{7311} + s_{8311}) \\
&\quad + (s_{4211} + s_{5211} + 2s_{6211} + s_{7211} + 2s_{8211} + s_{9211} + s_{(10,2,1,1)}) \\
&\quad + (s_{6111} + s_{7111} + 2s_{8111} + 2s_{9111} + 2s_{(10,1,1,1)} + s_{(11,1,1,1)} + s_{(12,1,1,1)}); \\
\mathcal{A}_7^{(5)} &= s_{31111} + s_{41111} + s_{51111} + s_{61111} + s_{71111}; \\
\mathcal{A}_7^{(6)} &= s_{111111}.
\end{aligned}$$

4.7. Asymptotics of coefficients. Another interesting feature of the \mathcal{E}_n is that there is an “asymptotic stability” of coefficients as n grows. Indeed, if we denote by $\tilde{\mu}$ the partition obtained by removing the first part of μ , we observe a stabilization of the first terms of

$$\tilde{\mathcal{E}}_n := \sum_{\mu \vdash n} \mathcal{C}_\mu \otimes s_{\tilde{\mu}},$$

as exhibited in the sequence

$$\begin{aligned}
\tilde{\mathcal{E}}_1 &= 1 \otimes 1, \\
\tilde{\mathcal{E}}_2 &= 1 \otimes 1 + s_1 \otimes s_1, \\
\tilde{\mathcal{E}}_3 &= 1 \otimes 1 + (s_1 + s_2) \otimes s_1 + (s_{11} + s_3) \otimes s_{11}, \\
\tilde{\mathcal{E}}_4 &= 1 \otimes 1 + (s_1 + s_2 + s_3) \otimes s_1 + (s_{11} + s_3 + s_{21} + s_{31} + s_4 + s_5) \otimes s_{11} \\
&\quad + (s_2 + s_{21} + s_4) \otimes s_2 + (s_{111} + s_{31} + s_{41} + s_6) \otimes s_{111}, \\
&\quad \vdots
\end{aligned}$$

In other words, the limit as n goes to infinity of $\tilde{\mathcal{E}}_n$ makes sense and we have

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{E}}_n = 1 \otimes 1 + \Omega \otimes s_1 + (\Omega \cdot \Omega_{\text{odd}} - \Omega_{\text{even}}) \otimes s_{11} + (\Omega \cdot \Omega_{\text{even}} + \Omega_{\text{even}}) \otimes s_2 + \dots \quad (4.18)$$

where we set $\Omega := \Omega_{\text{odd}} + \Omega_{\text{even}}$, with $\Omega_{\text{even}} := \sum_{k=1}^{\infty} s_{2k}$, and $\Omega_{\text{odd}} := \sum_{k=0}^{\infty} s_{2k+1}$. In fact, for all $n > 1$, we have $\tilde{\mathcal{E}}_{n-1}$ sitting inside⁹ $\tilde{\mathcal{E}}_n$. We may thus form the power series

$$\tilde{\mathcal{E}}_\infty(z) := \tilde{\mathcal{E}}_1 z + \sum_{n \geq 2} (\tilde{\mathcal{E}}_n - \tilde{\mathcal{E}}_{n-1}) z^n,$$

⁹This is to say that the difference $\tilde{\mathcal{E}}_n - \tilde{\mathcal{E}}_{n-1}$ is positive.

from which any individual \mathcal{E}_n can readily be obtained. One may obtain a z -analogue of (4.18) of the form

$$\begin{aligned}\tilde{\mathcal{E}}_\infty(z) &= z \otimes 1 + z \Omega(z) \otimes s_1 \\ &\quad + z(z \Omega(z) \cdot \Omega_{\text{odd}}(z) + z \Omega_{\text{odd}}(z) - \Omega(z)) \otimes s_{11} + \\ &\quad + z(z \Omega(z) \cdot \Omega_{\text{even}}(z) + z \Omega_{\text{even}}(z) + (z-1)\Omega(z) - (z-1)s_1) \otimes s_2 + \dots\end{aligned}$$

where we set $\Omega(z) := z \Omega_{\text{odd}}(z^2) + \Omega_{\text{even}}(z^2)$; with

$$\Omega_{\text{even}}(z) := \sum_{k=1}^{\infty} s_{2k} z^k, \quad \text{and} \quad \Omega_{\text{odd}}(z) := \sum_{k=0}^{\infty} s_{2k+1} z^k.$$

5. LINK TO DELTA OPERATORS, AND THE DELTA-CONJECTURE

5.1. Skewing versus Δ -operators. The ‘‘Delta-Conjecture’’ (see [17]) proposes an explicit combinatorial description for the evaluation at e_n of the Macdonald ‘‘eigenoperator’’ Δ'_{e_k} . These operators were originally introduced in [8], up to a slight shift in eigenvalues. We will not need the precise combinatorial statement of the Delta-Conjecture, since we only consider the symmetric function $\Delta'_{e_k}(e_n)$. Our discussion below leads to the conclusion that, for each μ , the coefficient $\langle \Delta'_{e_k}(e_n), s_\mu \rangle$ should be Schur positive¹⁰. This is a stronger positivity statement than that of the original Delta-Conjecture. The special case $\Delta_{e_2}(e_n)$ was in a sense previously considered in [26], not referring to a Schur expansion but showing an equivalent result.

Conjecture.1 states that $\Delta'_{e_k}(e_n)$ may be obtained simply by skewing first components in \mathcal{E}_n by the elementary symmetric function e_j , with $j = n - 1 - k$. Hence, we may consider $(e_j^\perp \otimes \text{Id}) \mathcal{E}_n$ as a multivariate analogue of $\Delta'_{e_k}(e_n)$. Implicitly, this implies that we have a $(\text{GL}_\infty \times \mathbb{S}_n)$ -module interpretation for $\Delta'_{e_k}(e_n)$. Once again we emphasize that the above conjecture could not have been stated in the original restricted context of two sets of variables, namely inside $\mathcal{R}_n^{(2)}$, in which the Δ_{e_k} operators are usually considered. It has recently been shown (see [30]) that Conjecture.1 implies Conjecture.4. It also follows, together with (2.8), that we have the equality

$$\langle \Delta'_{e_k} e_n, s_{(n-j, 1^j)} \rangle = \langle \Delta'_{e_j} e_n, s_{(n-k, 1^k)} \rangle, \quad (5.1)$$

for all k and j . This follows directly using known identities involving Macdonald polynomials, thus giving indirect support to (1.9). Keeping the same convention for j and k , we have

$$(e_j^\perp \otimes \text{Id}) \mathcal{E}_n(q, 1/q; \mathbf{x}) = \sum_{j=0}^k (-1)^{k-j} \frac{e_j[[n]_q]}{q^{j(n-1)}} \frac{e_n[[j+1]_q \mathbf{x}]}{[j+1]_q} \quad (5.2)$$

5.2. Inferred components of \mathcal{E}_n . Just as before, we consider the decomposition of \mathcal{E}_n into its length components $\mathcal{E}_n^{(k)}$, and the associate reduced-length components $\varepsilon_n^{(j)}$ (see (4.6)). Clearly, we have the bijective correspondence $\varepsilon_n^{(j)} \leftrightarrow \mathcal{E}_n^{(j)} = (\uparrow_{1^j} \otimes \text{Id}) \varepsilon_n^{(j)}$. One may see that the length of $\varepsilon_n^{(j)}$ (that is the maximal length of one of its GL_∞ -coefficients) is equal to $\min(j, n-1-j)$.

¹⁰Hence the same holds true for $\Delta_{e_k}(e_n)$.

Together with this length bound, Conjecture.1 implies that we may calculate the various $\varepsilon_n^{(j)}$, for all $0 \leq j < n \leq 6$, just from the knowledge of the length 2 expressions $\Delta'_{e_k} e_n$. As a matter of fact, for all n , we have the general formulas:

$$\varepsilon_n^{(0)} = 1 \otimes s_n, \quad (5.3)$$

$$\varepsilon_n^{(1)} = (e_1^\perp \otimes \text{Id}) H_n, \quad (5.4)$$

$$\varepsilon_n^{(2)} = (e_2^\perp \otimes \text{Id}) \nabla(e_n), \quad (5.5)$$

$$\varepsilon_n^{(n-3)} = \Delta'_{e_2}(e_n) + (s_1 + s_2) \otimes e_n + (1 \otimes e_{n-1}e_1) - \sum_{k=1}^n s_{k-1} \otimes e_k e_{n-k}. \quad (5.6)$$

$$\varepsilon_n^{(n-2)} = \sum_{k=1}^n s_{k-1} \otimes e_k e_{n-k} - (1 + s_1) \otimes e_n \quad (5.7)$$

$$\varepsilon_n^{(n-1)} = 1 \otimes e_n. \quad (5.8)$$

6. THE e -POSITIVITY PHENOMENON

As discussed in [4], most of the symmetric functions constructed via the elliptic Hall algebra approach exhibit an e -positivity when specialized at $t = 1$. We discuss here the case of \mathcal{E}_n , for which we get the specialization of any one of the parameters q_i to the value 1 via the plethystic evaluation at $\mathbf{q} + 1$ of the GL_∞ -coefficients \mathbf{c}_μ of \mathcal{E}_n . For the sake of discussion, let us set

$$\mathcal{F}_n := \mathcal{E}_n[\mathbf{q} + 1; \mathbf{z}], \quad (6.1)$$

and write

$$\begin{aligned} \mathcal{F}_n &= \sum_{\mu \vdash n} \mathbf{c}_\mu[\mathbf{q} + 1] \otimes s_\mu(\mathbf{z}), \\ &= \sum_{\nu \vdash n} \mathfrak{d}_\nu \otimes e_\nu(\mathbf{z}), \end{aligned} \quad (6.2)$$

with \mathfrak{d}_ν the coefficients of $e_\nu(\mathbf{z})$ in \mathcal{F}_n . Then, as far as we can check experimentally, all of the \mathfrak{d}_ν are **Schur positive**. For instance, we have

$$\begin{aligned} \mathcal{F}_4 &= 1 \otimes e_{1111} + (3s_1 + 2s_2 + s_3) \otimes e_{211} + (s_{11} + s_2 + s_{21} + s_4) \otimes e_{22} \\ &\quad + (2s_{11} + s_{21} + 2s_3 + s_{31} + s_4 + s_5) \otimes e_{31} + (s_{111} + s_{31} + s_{41} + s_6) \otimes e_4. \end{aligned}$$

We may also write

$$\mathfrak{d}_\nu = \langle \mathcal{F}_n, f_\nu \rangle.$$

The \mathbf{c}_ν are related to the \mathfrak{d}_μ as follows

$$\mathbf{c}_\mu[\mathbf{q} + 1] = \sum_{\nu} K_{\mu'\lambda} \mathfrak{d}_\lambda, \quad (6.3)$$

where the $K_{\mu\lambda}$ are the usual Kostka numbers. There is a close ties between this e -positivity phenomenon, Identity (2.8), and related conjectures. To see this, we recall that the coefficient of e_n in the e -expansion of s_μ vanishes for all μ except hooks.; and is known to be equal to

$(-1)^k$ when $\mu = (k+1, 1^{n-k-1})$. Since the forgotten symmetric functions f_ν are dual to the e_ν , we may write this as

$$\langle s_\mu, f_{(n)} \rangle = \begin{cases} (-1)^k, & \text{if } \mu = (k+1, 1^{n-k-1}), \\ 0 & \text{otherwise.} \end{cases}$$

We may then calculate that

$$\begin{aligned} \mathfrak{d}_{(n)} &= \langle \mathcal{E}_n[\mathbf{q}+1; \mathbf{z}], f_{(n)} \rangle \\ &= \sum_{\mu \vdash n} \mathbf{c}_\mu[\mathbf{q}+1] \langle s_\mu, f_{(n)} \rangle \\ &= \left(\sum_{k=0}^{n-1} (-1)^k \mathbf{c}_{(k+1, 1^{n-k-1})} \right) [\mathbf{q}+1] \\ &= \left(\sum_{k \geq 0} (-1)^k e_k^\perp \mathcal{A}_n \right) [\mathbf{q}+1]. \end{aligned}$$

Recall that we have

$$f[\mathbf{q}-1] = \sum_{k \geq 0} (-1)^k e_k^\perp f(\mathbf{q}),$$

so that we may conclude the above calculation to get

$$\mathfrak{d}_{(n)} = (\mathcal{A}_n[\mathbf{q}-1])[\mathbf{q}+1] = \mathcal{A}_n, \quad (6.4)$$

which is Schur positive. To get more, let μ be any partition of n which is largest in dominance order among those such that $\mathbf{c}_\mu \neq 0$. Then, it is easy to see that

$$\mathfrak{d}_{(n)} = \mathbf{c}_{1^n}, \quad \text{and} \quad \mathfrak{d}_{\mu'} = \mathbf{c}_\mu[\mathbf{q}+1]; \quad (6.5)$$

and that $\mathfrak{d}_{\rho'} = 0$ whenever $\mathbf{c}_\rho = 0$. We thus automatically have Schur positivity in the above cases whenever \mathbf{c}_{1^n} and \mathbf{c}_μ are positive themselves. Experiments suggest that, when $m < n$, we have

$$\mathfrak{d}_{(k+1, 1^{n-k-1})} = \mathbf{c}_{(n-k, 1^k)}, \quad (6.6)$$

Further experiments also suggest that, when $m = n$ and $k < n-1$

$$\mathfrak{d}_{(n-k, 1^k)} = \sum_{j=1}^{k+1} \mathbf{c}_{(j, 1^{n-k-1})}, \quad (\text{setting } \mathfrak{d}_{(n-k, 1^k)} := \mathfrak{d}_{n, (n-k, 1^k)}). \quad (6.7)$$

Clearly, $\mathfrak{d}_{(1^n)} = 1$ by the above remark.

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Much of this work would not have been achieved without the possibility of perusing sufficiently large expressions resulting from difficult explicit computations. These calculations were very elegantly realized by Nicolas Thiéry using the Sage computer algebra system, together with an inspired use of the right mathematical properties of objects considered, and his special expertise of higher Specht modules. Indeed, direct calculations of the relevant symmetric function expressions rapidly become unfeasible, even with powerful computers.

Such explicit calculations must thus rely on an artful combination of high-level computer algebra skills, and a subtle understanding of the mathematical structures involved.

APPENDICES

Symmetric functions and plethysm. We mainly use Macdonald's notations (see [21]). Thus p_μ , e_μ , and h_μ respectfully stand for the **power-sum**, **elementary**, and **complete homogeneous** symmetric functions, with indices integer partitions $\mu = \mu_1\mu_2 \cdots \mu_k$ of n . Recall that these are symmetric functions that are all homogeneous of degree $n = |\mu| := \mu_1 + \mu_2 + \cdots + \mu_k$. Assuming that they are evaluated in enough variables, they respectively form bases of the homogenous degree n component Λ_n of the graded ring of symmetric function Λ . It is often useful to avoid writing variables, implicitly making the above assumption. The **length** $\ell(\mu)$ is equal to the number of non-zero parts (the μ_i) of μ . Recall that partitions are often described in terms of their **Ferrers diagram** (herein in French notation). This diagram is the set of (i, j) in $\mathbb{N} \times \mathbb{N}$ (considered as points in the usual cartesian coordinates), with $0 \leq i \leq \mu_{j+1} - 1$, for $0 \leq j < \ell(\mu)$. We say that (i, j) is a cell of μ , and often write $(i, j) \in \mu$ when the context makes this convention clear. The row lengths of this diagram are the parts of μ . The **conjugate** μ' of μ is the partition with diagram equal to the set $\{(j, i) \mid (i, j) \in \mu\}$. The **hook length** of a cell (i, j) of μ is defined as

$$h(i, j) = h_{ij} = \mu_{j+1} + \mu'_{i+1} - i - j - 1.$$

We set

$$\iota(\mu) := \#\{(i, j) \in \mu \mid i > j\}. \quad (6.8)$$

Among the usual bases of Λ_d , the most interesting is that of Schur functions s_μ . The skew-Schur functions may be considered as enumerators of semi-standard tableaux. In formula, this says that

$$s_{\lambda/\mu}(\mathbf{z}) = \sum_{\tau: \lambda/\mu \rightarrow \mathbb{N}} \mathbf{z}_\tau, \quad \text{with} \quad \mathbf{z}_\tau := \prod_{c \in \lambda/\mu} z_{\tau(c)},$$

with τ running over the set of semi-standard fillings of the cells $c = (i, j)$ of λ/μ . This is to say that, whenever it makes sense,

$$\tau(i, j) \leq \tau(i + 1, j), \quad \text{and} \quad \tau(i, j) < \tau(i, j + 1).$$

When μ is the empty partition, these correspond to the above Schur function. It follows that $s_\lambda(z_1, \dots, z_k) = 0$, whenever $k < \ell(\lambda)$, since no such filling exists.

The Schur functions are orthonormal for the usual **Hall scalar product** on Λ , which may be defined by setting

$$\langle p_\lambda, p_\mu \rangle := z_\mu \delta_{\lambda, \mu}.$$

The integer z_μ is such that $n!/z_\mu$ (for μ a partition of n) is equal to the cardinality of the conjugacy class of permutations characterized the fact that they all the same cycle structure given by the partition ν . For a given symmetric function f , the linear operator f^\perp is the **adjoint** to the linear operation of multiplication by f . In formula, we have that

$$\langle f \cdot g_1, g_2 \rangle = \langle g_1, f^\perp g_2 \rangle,$$

holds for any symmetric functions g_1 and g_2 . The classical (dual) Pieri rule (see [21]), implies that

$$e_k^\perp s_\mu = \sum_{\lambda \subset \mu} s_\lambda,$$

where the sum is over the partitions that can be obtained from μ by removing k cells, no two of which lying on the same row. In particular, $e_k^\perp s_\mu$ vanishes if μ has less than k rows.

Positive integral linear combination of Schur functions are said to be **Schur positive**. This notion is often extended to polynomials with coefficients in \mathbb{N} , in some set of variables (often q, t , or q_i). We also write $F \preceq G$ whenever the difference $G - F$ is Schur positive. It may easily be seen that this is an order relation, which is compatible with many of the operations on symmetric functions.

Informally speaking, the **plethysm** $f \circ g$ of two symmetric functions f and g is calculated by replacing the variables in f by the monomials that occur in g . This process may be recursively defined by the properties. In fact, this is a special case of a λ -ring calculations $f[A]$ (with $A = g$), in the context of which symmetric function may be considered as operators on elements A of the underlying ring. Rules of “plethystic evaluation” are as follows, assuming that α and β are scalars, and that A and B lie in some suitable ring:

- (1) $(\alpha f + \beta g)[A] = \alpha f[A] + \beta g[A]$,
- (2) $(f \cdot g)[A] = f[A] \cdot g[A]$,
- (3) $p_k[A \pm B] = p_k[A] \pm p_k[B]$,
- (4) $p_k[A \cdot B] = p_k[A] \cdot p_k[B]$,
- (5) $p_k[A/B] = p_k[A]/p_k[B]$,
- (6) $p_k[p_j] = p_{kj}$,
- (7) $p_k[x] = x^k$, whenever x a “variable”,
- (8) $p_k[c] = c$, whenever c a “constant”.

Hence we should always specify clearly what are variables and what are constants. The first two properties make it clear that any evaluation of the form $f[A]$ may be reduced to instances of the form $p_k[A]$. We also assume that property (2) extends to denumerable sums. See [6] for more on plethysm.

Macdonald polynomials, and operators. Recall that the set of **combinatorial Macdonald polynomials**¹¹ $\{H_\mu(q, t; \mathbf{z})\}_{\mu \vdash n}$ forms a linear basis of the ring $\Lambda(q, t)$ (of symmetric functions in the variables $\mathbf{z} = r_1, r_2, r_3, \dots$ over the field $\mathbb{Q}(q, t)$) uniquely characterized by the equations

- (i) $\langle s_\lambda(\mathbf{z}), H_\mu[q, t; (1 - q) \mathbf{z}] \rangle = 0$, if $\lambda \not\preceq \mu$,
- (ii) $\langle s_\lambda(\mathbf{z}), H_\mu[q, t; (1 - t) \mathbf{z}] \rangle = 0$, if $\lambda \not\preceq \mu'$, and
- (iii) $\langle s_n(\mathbf{z}), H_\mu(q, t; \mathbf{z}) \rangle = 1$,

¹¹Sometimes denoted by \tilde{H}_μ .

involving plethystic notation. See [18, Section 3.5] for more details on these dominance order triangularities. The operators ∇ and Δ'_{e_k} , introduced in [8], are special instances of **Macdonald eigenoperators**. This is to say that they have the Macdonald polynomials H_μ as joint eigenfunctions. Namely, we set

$$\Delta_{e_k}(H_\mu) := e_k[B_\mu] H_\mu, \quad \Delta'_{e_k}(H_\mu) := e_k[B_\mu - 1] H_\mu, \quad \text{with} \quad B_\mu := \sum_{(i,j) \in \mu} q^i t^j.$$

In the last expression, the sum runs over the cells of the partition μ , using Cartesian coordinates for these cells (occurring in the Ferrers diagram of ν , in French notation). Recall that the ‘‘plethystic notation’’ $e_k[B_\mu]$ simply means that the symmetric function e_k is evaluated in the ‘‘variables’’ $q^i t^j$. On homogeneous symmetric functions of degree n (and only for those), the operator ∇ coincides with $\Delta_{e_n} = \Delta'_{e_{n-1}}$. In other terms, the associated eigenvalues are

$$T_\mu := q^{\eta(\mu')} t^{\eta(\mu)} = \prod_{(i,j) \in \mu} q^i t^j, \quad (6.9)$$

where we set¹² $\eta(\mu) := \sum_k (k-1)\mu_k$, for $\mu = \mu_1 \cdots \mu_\ell$. Among the many interesting formulas pertaining to the H_μ , we have

$$\langle H_\mu, s_{n-k, 1^k} \rangle = e_k[B_\mu - 1], \quad \text{in particular} \quad \langle H_\mu, s_{1^n} \rangle = T_\mu. \quad (6.10)$$

Observe the special instance:

$$\langle H_n, s_{n-k, 1^k} \rangle = q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \quad (6.11)$$

We also have the symmetries

$$H_\mu(q, t; \mathbf{x}) = T_\mu \omega H_\mu(1/q, 1/t; \mathbf{x}), \quad (6.12)$$

$$H_\mu(t, q; \mathbf{x}) = H'_\mu(q, t; \mathbf{x}). \quad (6.13)$$

The Δ -conjecture. In [17], one finds an explicit combinatorial formula conjectured to be equal to $\Delta'_{e_k} e_n$. With our particular point of view, it takes the form

$$\Delta'_{e_k} e_n = \sum_{\mu \subseteq \delta_{nn}} \left(\sum_{\substack{J \supseteq \text{des}(\mu) \\ \#J=k}} q^{(J, \mathbf{a})} \right) \mathbb{L}_\mu(t; \mathbf{z}) \quad (6.14)$$

where the indices J (in the inner sum) run over all suitable subsets of $[n] := \{1, 2, \dots, n-1\}$, and (J, \mathbf{a}) is shorthand for $\sum_{i \in J} a_i$. We denote ger by $\mathbb{L}_\mu(t; \mathbf{z})$ the vertical strip LLT-polynomial associated to the path μ , See [13] for more on these, in particular for a proof that $\mathbb{L}(1+t; \mathbf{z})$ is e -positive. For instance, when $k=0$, the only non-zero term of the outer sum corresponds to $\mu=0$, for which $s_{(\mu+1^n)/\mu} = e_n$, thus agreeing with $\Delta'_{e_0} e_n$. At the opposite end of the spectrum, when $k=n-1$, there is but one term in the inner sum (since J must be equal to $[n]$) which is clearly equal to $q^{\text{area}(\mu)}$. Observe that, at $t=1$, the above expression simplifies to

$$\Delta'_{e_k} e_n|_{t=1} = \sum_{\mu \subseteq \delta_{nn}} \left(\sum_{\substack{J \supseteq \text{des}(\mu) \\ \#J=k}} q^{(J, \mathbf{a})} \right) s_{(\mu+1^n)/\mu}(\mathbf{z}). \quad (6.15)$$

¹²This is what Macdonald denotes by $n(\mu)$.

The e -expansions of \mathcal{F}_n .

$$\begin{aligned}
\mathcal{F}_1 &= 1 \otimes e_1, \\
\mathcal{F}_2 &= \mathcal{A}_2 \otimes e_2 + 1 \otimes e_{11}, \\
\mathcal{F}_3 &= \mathcal{A}_3 \otimes e_3 + (2s_1 + s_2) \otimes e_{21} + 1 \otimes e_{111}, \\
\mathcal{F}_4 &= \mathcal{A}_4 \otimes e_4 + (e_1^\perp \mathcal{A}_4 + \mathcal{A}_3) \otimes e_{31} \\
&\quad + (s_{11} + s_{21} + s_2 + s_4) \otimes e_{22} + (3s_1 + 2s_2 + s_3) \otimes e_{211} + 1 \otimes e_{1111}, \\
\mathcal{F}_5 &= \mathcal{A}_5 \otimes e_5 + (e_1^\perp \mathcal{A}_5 + \mathcal{A}_4) \otimes e_{41} + (2s_{111} + 2s_{211} + s_{311} + s_{22} + s_{32} + s_{42} \\
&\quad + 2s_{21} + 2s_{31} + 3s_{41} + 2s_{51} + s_{61} + 2s_4 + s_6 + s_7 + s_8) \otimes e_{32} \\
&\quad + (e_2^\perp \mathcal{A}_5 + e_1^\perp \mathcal{A}_4 + \mathcal{A}_3) \otimes e_{311} \\
&\quad + (s_{22} + 3s_{11} + 4s_{21} + 2s_{31} + s_{41} + 3s_2 + 2s_3 + 2s_4 + 2s_5 + s_6) \otimes e_{221} \\
&\quad + (4s_1 + 3s_2 + 2s_3 + s_4) \otimes e_{2111} + 1 \otimes e_{11111}, \\
\mathcal{F}_6 &= \mathcal{A}_6 \otimes e_6 + (e_1^\perp \mathcal{A}_6 + \mathcal{A}_5) \otimes e_{51} \\
&\quad + (2s_{1111} + 2s_{2111} + s_{3111} + s_{4111} + s_{431} + s_{221} + 2s_{321} + 3s_{421} + s_{521} + s_{621} \\
&\quad + 2s_{211} + 2s_{311} + 5s_{411} + 3s_{511} + 4s_{611} + s_{711} + s_{811} \\
&\quad + s_{44} + s_{54} + 2s_{43} + 2s_{53} + s_{63} + s_{73} \\
&\quad + 2s_{32} + 3s_{42} + 3s_{52} + 3s_{62} + 3s_{72} + s_{82} + s_{92} \\
&\quad + 2s_{41} + 2s_{51} + 3s_{61} + 3s_{71} + 3s_{81} + 3s_{91} + s_{(10,1)} + s_{(11,1)} \\
&\quad + 2s_7 + s_9 + 2s_{(11)} + s_{(13)}) \otimes e_{42} \\
&\quad + (e_2^\perp \mathcal{A}_6 + e_1^\perp \mathcal{A}_5 + \mathcal{A}_4) \otimes e_{411} \\
&\quad + (s_{1111} + s_{2111} + s_{3111} + s_{331} + s_{221} + s_{321} + s_{421} + s_{521} \\
&\quad + s_{211} + 2s_{311} + 2s_{411} + 2s_{511} + s_{611} + s_{711} \\
&\quad + s_{44} + s_{33} + s_{43} + s_{53} + s_{63} \\
&\quad + s_{22} + 2s_{42} + 2s_{52} + s_{62} + s_{72} + s_{82} \\
&\quad + 2s_{41} + s_{51} + s_{61} + 2s_{71} + s_{81} + s_{91} + s_{(10,1)} + s_6 + s_9 + s_{(12)}) \otimes e_{33} \\
&\quad + (2s_{221} + 3s_{321} + s_{421} + 6s_{111} + 8s_{211} + 8s_{311} + 4s_{411} + 3s_{511} + s_{611} \\
&\quad + 2s_{33} + 3s_{43} + s_{53} + 4s_{22} + 8s_{32} + 8s_{42} + 5s_{52} + 4s_{62} + s_{72} \\
&\quad + 6s_{21} + 10s_{31} + 12s_{41} + 12s_{51} + 10s_{61} + 5s_{71} + 4s_{81} \\
&\quad + 6s_4 + 4s_5 + 4s_6 + 4s_7 + 6s_8 + 2s_9 + s_{91} + 3s_{(10)} + s_{(11)}) \otimes e_{321} \\
&\quad + (e_3^\perp \mathcal{A}_6 + e_2^\perp \mathcal{A}_5 + e_1^\perp \mathcal{A}_4 + \mathcal{A}_3) \otimes e_{3111} \\
&\quad + (s_{221} + s_{111} + 2s_{211} + s_{311} + s_{411} + 2s_{22} + s_{32} + s_{42} + s_{52} \\
&\quad + 2s_{21} + 2s_{31} + 2s_{41} + 2s_{51} + s_{61} + s_{71} + s_3 + 2s_5 + s_7 + s_9) \otimes e_{222} \\
&\quad + (3s_{22} + 2s_{32} + s_{42} + 6s_{11} + 9s_{21} + 7s_{31} + 5s_{41} + 2s_{51} + s_{61} \\
&\quad + 6s_2 + 6s_3 + 6s_4 + 4s_5 + 5s_6 + 2s_7 + s_8) \otimes e_{2211} \\
&\quad + (5s_1 + 4s_2 + 3s_3 + 2s_4 + s_5) \otimes e_{21111} + 1 \otimes e_{111111}
\end{aligned}$$

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