AN UPPER BOUND PROBLEM FOR TRIANGULATIONS OF MANIFOLDS

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1. Introduction

The study of face numbers of simplicial complexes using Stanley–Reisner rings is one of major research topics in algebraic combinatorics. The starting point of this research is Stanley's Upper Bound Theorem (UBT for short) for spheres, which gives sharp upper bounds of face numbers of triangulated spheres for a given number of vertices. Since then, the idea of Stanley–Reisner rings has been applied to obtain a number of interesting results on face numbers including a complete characterization of face numbers (g-theorem) for simplicial polytopes [BL, St80] and triangulated spheres [Adi].

One interesting research topic on this subject is to extend results on face numbers of triangulations of spheres, such as the UBT and the g-theorem, to more general (closed) manifolds. One possible extension of the UBT would be to find an upper bound which holds for triangulations of all manifolds. See [No] for results on this direction. The other possible extension of the UBT is to find a sharp upper bound of face numbers of triangulations for each fixed manifold, but, while the UBT for spheres is a basic result in algebraic combinatorics, we seem to miss a good method to obtain such a result for manifolds which are not spheres. The aim of this article is to pose a problem which hopefully initiate to study such upper bounds.

We first formulate a problem. We say that a simplicial complex Δ is a **triangulation** of a topological space X if its geometric realization $|\Delta|$ is homeomorphic to X. Let $f_i(\Delta)$ be the number of the i-dimensional faces of Δ. The vector $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_d(\Delta))$, where d is the dimension of Δ and $f_{-1}(\Delta) = 1$, is called the ***f*-vector** of Δ. We are interested in the following combinatorial invariant of a topological manifold M:

$$f_i^{\max}(M,n) = \max\{f_i(\Delta) \mid \Delta \text{ is an } n \text{ vertex triangulation of } M\},$$

where n is assumed to be larger than or equal to $f_0^{\min}(M)$ the minimal number of vertices which is required to triangulate M. The UBT for spheres can be considered as a result determining $f_i^{\max}(\mathbb{S}^{d-1}, n)$. Indeed, it tells that $f_i^{\max}(\mathbb{S}^{d-1}, n)$ equals to the f_i of a cyclic d-polytope with n vertices. Using this notation, our question can be stated as follows:

Question 1.1. Can we determine $f_i^{\max}(M,n)$ for all i and $n \geq f_0^{\min}(M)$ for (some classes of) closed manifolds M?

It is probably not tractable to get a complete answer to the above question for arbitrary manifold. Indeed, to get a complete answer, we need to know the number $f_0^{\min}(M)$ but in general it is a very difficult problem to determine this number (see [Lu]). On the other hand, it is still an interesting problem to study the above question for particular classes of manifolds, and probably the first case which we should understand is the case when M is a sphere bundle over the circle.

One reason to consider sphere bundles over the circle is that we have enough good information on lower bounds of f-vectors of their triangulations. There are only two topological types of $\mathbb{S}^k$-bundles over $\mathbb{S}^1$. One of them is the product $\mathbb{S}^k \times \mathbb{S}^1$ and the other one is a non-orientable $\mathbb{S}^k$-bundle $\mathbb{S}^k \underline{\times} \mathbb{S}^1$, called a *generalized Klein bottle* (see [Ste]). Every triangulation of an $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$ has at least $2d+1$ vertices [BK], and there is only one $2d+1$ vertex triangulation whose topological type is $\mathbb{S}^{d-2} \times \mathbb{S}^1$ when d is odd and $\mathbb{S}^{d-2} \underline{\times} S^1$ when d is even [BD, CSS]. Moreover, it is known that stacked triangulations gave a sharp lower bound of f-vectors of their triangulations for a fixed number of the vertices [BD, Theorem 3.12]. Considering these known results, our problem can be formulated as follows.

Problem 1.2. Determine $f_i^{\max}(\mathbb{S}^k \times \mathbb{S}^1, n)$ and $f_i^{\max}(\mathbb{S}^k \underline{\times} \mathbb{S}^1, n)$ for all i and $n \geq 2d+2$.

Here are some partial known results. There is a generalization of the UBT given by Novik [No] which guarantees that $f_i^{\max}(\mathbb{S}^{d-2} \times \mathbb{S}^1, n)$ and $f_i^{\max}(\mathbb{S}^{d-2} \underline{\times} \mathbb{S}^1, n)$ are bounded above by the f-vector of a cyclic d-polytope with n vertices, but this bound is not sharp. Also, Chestnut, Sapir and Swartz [CSS, Theorem 4.1] gave a complete characterization of all possible pairs (f_0, f_1) for each $\mathbb{S}^{d-2} \times \mathbb{S}^1$ and $\mathbb{S}^{d-2} \underline{\times} \mathbb{S}^1$, which gives an answer to Problem 1.2 for $i=1$. However, even a reasonable conjecture for precise values of $f_i^{\max}(\mathbb{S}^{d-2} \times \mathbb{S}^1, n)$ and $f_i^{\max}(\mathbb{S}^{d-2} \underline{\times} \mathbb{S}^1, n)$ are not known when $i \geq 2$.

In this note, we present some recent algebraic results which might help to solve this upper bound problem for manifolds, and, for sphere bundles over the circle, we present in Problem 3.3 conjectural upper bounds of face numbers of triangulations of these manifolds which is expected to be close to an actual values of $f_i^{\max}$.

This note is organized as follows. In section 2, we recall the UBT for spheres and its connection to Stanley–Reisner ring theory. In section 3, we pose our main conjectural upper bounds. This bounds use h''-vectors and $\widetilde{g}$-vectors, which appear in recent studies of face numbers of triangulated manifolds, and we explain algebraic meanings of these vectors in section 4. In section 5, we present an example of a triangulation of $\mathbb{S}^5 \underline{\times} \mathbb{S}^1$ whose face vector attains our conjectural upper bounds. In section 6, we discuss some more problems relating to the contents of this article.

2. Background: The UBT and Stanley–Reisner rings

Before discussing face numbers of manifolds, we recall the classical UBT for spheres and its connection to Stanley–Reisner rings. An algebraic idea to prove the UBT will be used to formulate our conjectural upper bounds of face numbers of triangulations of sphere bundles over the circle. We refer the readers to [Sta96, St14] for a history and a background on the UBT.

The Upper Bound Theorem. We first define necessary notation. Let Δ be a finite abstract simplicial complex. An element F of Δ is called a **face** of Δ, and the **dimension** of a face F is $\#F-1$, where $\#W$ denotes the cardinality of a finite set W. The **dimension** of a simplicial complex Δ is the maximum of the dimension of its faces. Recall that $f(\Delta)=(f_{-1}(\Delta),\dots,f_{d-1}(\Delta))$ is the f-vector of a $(d-1)$-dimensional simplicial complex Δ. We write $C(n,d)$ for the boundary complex of a cyclic d-polytope with n vertices (that is, the convex hull of n distinct points on the moment curve $\{(t,t^2,\dots,t^d)\in\mathbb{R}^d \mid t\in\mathbb{R}\}$). Below is the UBT for spheres.

Theorem 2.1 (UBT for spheres)**.** *If Δ is an n vertex triangulation of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$, then $f_i(\Delta)\le f_i(C(n,d))$ for all $i=0,1,\dots,d-1$.*

While we write the UBT in terms of f-vectors to see that it determines $f_i^{\max}(\mathbb{S}^d,n)$ (indeed it tells $f_i^{\max}(\mathbb{S}^{d-1},n)=f_i(C(n,d))$), it is more convenient to state the UBT using another vectors known as h- and g-vectors. The ***h*-vector** of a $(d-1)$-dimensional simplicial complex Δ is the vector $h(\Delta)=(h_0(\Delta),h_1(\Delta),\dots,h_d(\Delta))$ defined by $h_i(\Delta)=\sum_{k=0}^{i}(-1)^{i-k}\binom{d-k}{d-i}f_{k-1}(\Delta)$. Also, if Δ is a triangulation of $\mathbb{S}^{d-1}$, then its ***g*-vector** is the vector $g(\Delta)=(g_0(\Delta),g_1(\Delta),\dots,g_{\lfloor\frac{d}{2}\rfloor}(\Delta))$ defined by $g_i(\Delta)=h_i(\Delta)-h_{i-1}(\Delta)$ for $i=0,1,\dots,\lfloor\frac{d}{2}\rfloor$, where $h_{-1}(\Delta)=0$. Below are some remarks on h- and g-vectors.

- We have $f_{i-1}=\sum_{k=0}^{i}\binom{d-k}{d-i}h_k(\Delta)$. In particular, knowing $f(\Delta)$ is equivalent to knowing $h(\Delta)$ (if we know the dimension of Δ).
- Suppose that Δ is a triangulation of the $(d-1)$-sphere $\mathbb{S}^{d-1}$. Then $h(\Delta)$ is non-negative, unimodal and symmetric. In particular, the symmetry tells that knowing $h(\Delta)$ is equivalent to knowing $g(\Delta)$.
- Bounds for h-vectors induces bounds for f-vectors. Also, bounds for g-vectors induces bounds for h-vectors.

It is known that the h-vectors of cyclic polytopes are given by $h_k(C(n,d))=\binom{n-d+k-1}{k}$ for $k\le\frac{d}{2}$. The UBT is actually proved by proving the following statement.

Theorem 2.2 (UBT, h-version)**.** *If Δ is an n vertex triangulation of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$, then*

$$h_k(\Delta)\le\binom{n-d+k-1}{k}\quad\text{for } k\le\tfrac{d}{2}.$$

Also, the following g-vector version of the statement also holds.

Theorem 2.3 (UBT, g-version)**.** *If Δ is an n vertex triangulation of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$, then*

$$g_k(\Delta)\le\binom{n-d+k-2}{k}\quad\text{for } k\le\tfrac{d}{2}.$$

We note that the g-version implies the h-version, and the h-version implies the f-version of the UBT.

Algebraic meanings of the h- and g-version. We explain algebraic meanings of h- and g-version of the UBT using Stanley–Reisner rings. Let Δ be a simplicial complex with the vertex set $[n]=\{1,2,\dots,n\}$ and $S=\mathbb{F}[x_1,\dots,x_n]$ the polynomial ring over an infinite field $\mathbb{F}$. The ideal $I_\Delta=(x^F\mid F\notin\Delta)$, where $x^F=\prod_{i\in F}x_i$, is called the **Stanley–Reisner ideal** and the ring $\mathbb{F}[\Delta]=S/I_\Delta$ is called the **Stanley–Reisner ring**. It is known that for any $(d-1)$-dimensional simplicial complex Δ, there is a sequence of linear forms $\Theta=\theta_1,\dots,\theta_d\in S$ such that $\mathbb{F}[\Delta]/(\Theta\mathbb{F}[\Delta])$ is Artinian. Such a sequence Θ is called a **linear system of parameters** (l.s.o.p. for short) for $\mathbb{F}[\Delta]$. For a homogenous ideal I, the formal power series $\mathrm{Hilb}(S/I,t)=\sum_{k\ge0}\dim_{\mathbb{F}}(S/I)_k t^k$ is called the **Hilbert series** of S/I, where M_k denotes the degree k homogeneous component of a graded S-module M.

Now, assume that Δ is a triangulation of $\mathbb{S}^{d-1}$. It is known that $\mathbb{F}[\Delta]$ is Cohen–Macaulay, which means that for any (equivalently, some) l.s.o.p. Θ for $\mathbb{F}[\Delta]$ one has

$$\text{(1)} \qquad \mathrm{Hilb}\big(\mathbb{F}[\Delta]/(\Theta\mathbb{F}[\Delta]),t\big) = \textstyle\sum_{k=0}^{d} h_k(\Delta)t^k.$$

Also, since $\mathbb{F}[\Delta]/(\Theta\mathbb{F}[\Delta]) \cong S/(I_\Delta + (\Theta))$, we have the inequality

$$\text{(2)} \qquad \dim_{\mathbb{F}} \big(\mathbb{F}[\Delta]/(\Theta\mathbb{F}[\Delta])\big)_k \leq \dim_{\mathbb{F}}(S/(\theta_1,\dots,\theta_d))_k = \dim_{\mathbb{F}}(\mathbb{F}[x_1,\dots,x_{n-d}])_k.$$

Considering (1), one can see that the inequality (2) for $k = 0, 1, \dots, d$ is nothing but the h-version of the UBT (Theorem 2.2) since $\dim_{\mathbb{F}}(\mathbb{F}[x_1,\dots,x_{n-d}])_k = \binom{n-d+k-1}{k}$.

Similarly, the g-version has the following algebraic meaning. An algebraic g-theorem for spheres [Adi] (a shorter proof in characteristic 2 can be found in [APP, PP]) tells that, if Δ is a triangulation of $\mathbb{S}^{d-1}$ and if $\Theta = \theta_1,\dots,\theta_{d+1} \in S_1$ is sufficiently general, then $\theta_1,\dots,\theta_d$ is an l.s.o.p. for $\mathbb{F}[\Delta]$ and the multiplication

$$\times\theta_{d+1} : \big(\mathbb{F}[\Delta]/(\theta_1,\dots,\theta_d)\mathbb{F}[\Delta]\big)_{k-1} \to \big(\mathbb{F}[\Delta]/(\theta_1,\dots,\theta_d)\mathbb{F}[\Delta]\big)_k$$

is injective for $k \leq \frac{d+1}{2}$ and is surjective for $k \geq \frac{d+1}{2}$, which in particular tells

$$\text{(3)} \qquad \mathrm{Hilb}\big(\mathbb{F}[\Delta]/(\Theta\mathbb{F}[\Delta]),t\big) = \textstyle\sum_{k=0}^{\lfloor \frac{d}{2}\rfloor} g_k(\Delta)t^k.$$

Then the inequality

$$\text{(4)} \qquad \dim_{\mathbb{F}} \big(\mathbb{F}[\Delta]/(\Theta\mathbb{F}[\Delta])\big)_k \leq \dim_{\mathbb{F}}(S/(\theta_1,\dots,\theta_{d+1}))_k = \dim_{\mathbb{F}}(\mathbb{F}[x_1,\dots,x_{n-d-1}])_k$$

coincides with the g-version of the UBT (Theorem 2.3)[1].

3. Main problem

Now we present the main problem in this article. Recall that the g-version of the UBT can be expressed as follows.

Theorem 3.1 (UBT, g-version). *If Δ is an n vertex triangulation of the $(d-1)$-dimensional sphere $\mathbb{S}^{d-1}$ and $R = \mathbb{F}[x_1,\dots,x_{n-d-1}]$, then*

$$g_k(\Delta) \leq \dim_{\mathbb{F}} R_k \quad \textit{for } k \leq \tfrac{d}{2}.$$

Our formulation of the problem is based on this algebraic expression of the UBT.

To state the problem, we need h''- and $\widetilde{g}$-vectors, which are considered to be manifold versions of h- and g-vectors. To simplify the argument, we assume that $\mathbb{F}$ has characteristic 2 in the rest of this note[2]. For a simplicial complex Δ, we write $\widetilde{H}_i(\Delta;\mathbb{F})$ for the ith reduced homology group of Δ with coefficients in $\mathbb{F}$ and $\beta_i(\Delta) = \dim_{\mathbb{F}} \widetilde{H}_i(\Delta;\mathbb{F})$. For a $(d-1)$-dimensional simplicial complex Δ, we define the vectors $h'(\Delta) = (h'_0(\Delta), h'_1(\Delta),\dots,h'_d(\Delta))$ and $h''(\Delta) = (h'_0(\Delta), h'_1(\Delta),\dots,h'_d(\Delta))$ by

$$h'_i(\Delta) = h_i(\Delta) - \binom{d}{i}\left(\sum_{k=1}^{i-1}(-1)^{i-k}\beta_{k-1}(\Delta)\right)$$

[1] The algebraic g-theorem is not necessary to prove Theorem 2.3 since in the equality (3) the left-hand side is larger than or equal to the right-hand side even without the algebraic g-theorem, but (3) clarifies an algebraic meaning of g-vectors.

[2] This is assumed to ignore an orientability issue.

and

$$h_i''(\Delta) = \begin{cases} h_i'(\Delta) - \binom{d}{i}\beta_{i-1}(\Delta), & (i \neq d), \\ h_d'(\Delta), & (i = d). \end{cases}$$

When Δ is a triangulation of a connected closed $(d-1)$-manifold, we also define the vector $\widetilde{g}(\Delta) = (\widetilde{g}_0(\Delta), \widetilde{g}_1(\Delta), \dots, \widetilde{g}_{\lfloor \frac{d}{2} \rfloor}(\Delta))$ by

$$\begin{aligned} \widetilde{g}_i(\Delta) &= h_i(\Delta) - h_{i-1}(\Delta) - \binom{d+1}{i}\left(\sum_{k=1}^{i}(-1)^{i-k}\beta_{k-1}(\Delta)\right) \\ &= h_i'(\Delta) - h_{i-1}''(\Delta) - \binom{d+1}{i}\beta_{i-1}(\Delta). \end{aligned}$$

Similar to h-vectors and g-vectors, the above vectors have nice algebraic meanings in terms of Stanley–Reisner rings, and $\widetilde{g}$-vectors of triangulated closed manifolds are considered to play a role of g-vectors of triangulated spheres. Indeed, the following generalization of Theorem 3.1 is known.

Theorem 3.2. *Let $d \geq 4$, Δ an n vertex triangulation of a connected closed $(d-1)$-manifold, and $R = \mathbb{F}[x_1, \dots, x_{n-d-1}]$. Then there is an ideal $I \subset R$ having $\binom{d+1}{2}\beta_1(\Delta)$ generators of degree 2 such that*

$$\widetilde{g}_k(\Delta) \leq \dim_{\mathbb{F}}(R/I)_k \quad \textit{for} \quad k \leq \tfrac{d}{2}.$$

We will explain meanings of h'-, h''- and $\tilde{g}$-vectors as well as a more general statement which proves Theorem 3.2 in the next section.

Inspired from Theorem 3.2, we pose the following more explicit conjectural upper bounds for triangulations of sphere bundles over the circle.

Problem 3.3. Let $d \geq 4$, $n \geq 2d+1$ and $R = \mathbb{F}[x_1, \dots, x_{n-d-1}]$. Is it true that if Δ is an n vertex triangulation of an $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$, then

$$\widetilde{g}_k(\Delta) \leq \dim_{\mathbb{F}}(R/(x_1, \dots, x_d)^2)_k \quad \text{for} \quad k \leq \tfrac{d}{2}?$$

So the problem asks if we can take I in Theorem 3.2 as the ideal $(x_1, \dots, x_d)^2$. At this moment, it is not so clear if the Hilbert series of $R/(x_1, \dots, x_d)^2$ is really a right choice for the maximum of the $\widetilde{g}$-vectors in Problem 3.3. This choice is based on the following result in [Mu].

Theorem 3.4. *Let $d \geq 4$, $n \geq 2d+1$ and $R = \mathbb{R}[x_1, \dots, x_{n-d-1}]$. There is a triangulation of an $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$ such that*

$$\widetilde{g}_k(\Delta) = \dim_{\mathbb{F}}(R/(x_1, \dots, x_d)^2)_k \quad \textit{for} \quad k = 0, 1, \dots, \lfloor \tfrac{d-1}{2} \rfloor. \tag{5}$$

Remark 3.5. To study the problem it might be convenient to write the inequality in Problem 3.3 more explicitly. If Δ is a triangulation of an $\mathbb{S}^{d-2}$-bundle over $\mathbb{S}^1$, then it has one dimensional homology in dimensions 1 and $d-2$ so $\tilde{g}_0(\Delta) = 1$, $\tilde{g}_1(\Delta) = g_1(\Delta)$ and

$$\widetilde{g}_k(\Delta) = g_k(\Delta) - (-1)^k\binom{d+1}{k} \quad \text{for } k \geq 2.$$

Also, we have

$$\dim_{\mathbb{F}}(R/(x_1, \dots, x_d)^2)_k = \binom{n-2d+k-2}{k} + d\binom{n-2d+k-3}{k-1}$$

for $k \geq 2$ since as vector spaces we have an isomorphism (as vector spaces)

$$R/(x_1,\dots,x_d)^2 \cong \mathbb{F}[x_{d+1},\dots,x_{n-d-1}] \bigoplus (\oplus_{k=1}^{d} x_k \mathbb{F}[x_{d+1},\dots,x_{n-d-1}]).$$

Remark 3.6. The topological type of Δ in Theorem 3.4 given in [Mu] is $\mathbb{S}^{d-2}\times\mathbb{S}^1$ when d is even and $\mathbb{S}^{d-2}\underline{\times}\,\mathbb{S}^1$ when d is odd. We are not sure if the existence of triangulations satisfying (5) depends on a topological type.

4. Algebraic meanings of h''- and $\widetilde{g}$-vectors

In this section, we explain algebraic meanings of h''- and $\widetilde{g}$-vectors, in particular, explain how Theorem 3.2 follows from known results. For further results and backgrounds on these vectors, see the survey article [KN].

For a graded S-module M, its **socle** is the submodule $\mathrm{Soc}(M)=\{f\in M \mid \mathfrak{m}f=0\}$, where $\mathfrak{m}=(x_1,\dots,x_n)$ is the graded maximal ideal of S. Below is an algebraic meaning of h'- and h''-vectors.

Theorem 4.1. *Let Δ be a triangulation of a connected closed $(d-1)$-manifold with the vertex set $[n]$, Θ an l.s.o.p. for $\mathbb{F}[\Delta]$ and $R=\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]$. Then*

(1) (*Schenzel's formula*) $\mathrm{Hilb}(R,t)=\sum_{k=0}^{d} h'_k(\Delta)t^k$.
(2) (*Novik–Swartz*) *Let* $N=\bigoplus_{k=1}^{d-1}\mathrm{Soc}(R)_k$. *Then*
  (a) $\mathrm{Hilb}(N,t)=\sum_{k=1}^{d-1}\binom{d}{k}\beta_{k-1}(\Delta)t^k$,
  (b) $\mathrm{Hilb}(R/N,t)=\sum_{k=0}^{d} h''_k(\Delta)t^k$, *and*
  (c) *R/N is Gorenstein, in particular,* $h''_i(\Delta)=h''_{d-i}(\Delta)$ *for all* i.

The statement (1) is due to Schenzel [Sc] while (2) appear in [No, NS1, NS2]. We next state an algebraic meaning of $\widetilde{g}$-vectors, which essentially appeared in [APP] and in [MN].

Theorem 4.2. *Let Δ be a triangulation of a connected closed $(d-1)$-manifold. Let $\Theta=\theta_1,\dots,\theta_{d+1}\in S_1$ be general linear forms and $R'=\mathbb{F}[\Delta]/\Theta\mathbb{F}[\Delta]$. Then*

(1) $\dim_{\mathbb{F}} R'_k = h'_k(\Delta)-h''_{k-1}(\Delta)$ *for* $k\leq \frac{d+1}{2}$.
(2) *There is an ideal J of R' satisfying the following conditions*
  (a) $\dim_{\mathbb{F}} J_k=\binom{d+1}{k}\beta_{k-1}(\Delta)$ *for* $k\leq\frac{d}{2}$, *and*
  (b) $\mathrm{Hilb}(R'/J,t)=\sum_{k=0}^{\lfloor\frac{d}{2}\rfloor}\widetilde{g}_k(\Delta)t^k$.

Proof Sketch. Let R and N be as in Theorem 4.1. We recall the following known facts.

(I) The multiplication $\times\theta_{d+1}: R_{k-1}\to R_k$ is surjective for $k\geq\frac{d}{2}+1$.
(II) The multiplication $\times\theta_{d+1}:(R/N)_{k-1}\to(R/N)_k$ is injective for $k\leq\frac{d+1}{2}$ and is surjective for $k\geq\frac{d+1}{2}$.

The statement (I) is a consequence of [Sw, Theorem 2.6] and the algebraic g-theorem [Adi]. Also, the statement (II) appears in [APP].

Now the injectivity in (II) tells that, for $k\leq(d+1)/2$, the kernel of

$$\times\theta_{d+1}: R_{k-1}\to R_k$$

is N_{k-1}, so $\dim(R')_k=h'_k(\Delta)-h''_{k-1}(\Delta)$ proving (1).

We now consider (2). By (1) it suffices to prove (b). By taking the Matlis dual of the short exact sequence $0\to N\to R\to R/N\to 0$, we have the exact sequence

$$0\longrightarrow (R/N)^\vee \stackrel{\phi}{\longrightarrow} R^\vee \longrightarrow N^\vee\longrightarrow 0$$

where $M^\vee$ denotes the Matlis dual of a graded S-module M. Since $\theta_{d+1}N = 0$, we have $\theta_{d+1}N^\vee = 0$, which tells that $\theta_{d+1}R^\vee$ is a submodule of $\mathrm{Im}\phi$. Since R/N is Gorenstein, we have $R/N \cong (R/N)^\vee \cong \mathrm{Im}\phi$ if we shift the grading by d. Then, since $\phi(\theta_{d+1}(R/N)^\vee) \subset \theta_{d+1}R^\vee$,

$$\mathrm{Im}\phi/(\theta_{d+1}R^\vee) \cong R'/J$$

for some ideal J of R'. We prove that J satisfies the desired conditions.

The surjectivity (I) tells that $\times\theta_{d+1} : R^\vee_{-k} \to R^\vee_{-k+1}$ is injective for $\frac{d}{2}+1 \leq k \leq d$, so

$$\dim_{\mathbb{F}}(\theta_{d+1}R^\vee)_{-k+1} = \dim_{\mathbb{F}}(R^\vee)_{-k} = \dim_{\mathbb{F}} R_k \quad \text{ for } k \geq d/2+1.$$

Recall $h''_k(\Delta) = h''_{d-k}(\Delta)$ and $\beta_{k-1}(\Delta) = \beta_{d-k}(\Delta)$ for all k. Then, for $k \leq \frac{d}{2}$, since $R/N \cong \mathrm{Im}\phi$, we have

$$\begin{aligned}
\dim(\mathrm{Im}\phi/\theta_{d+1}R^\vee)_{-d+k} &= \dim(R/N)^\vee_{-d+k} - \dim(\theta_{d+1}R^\vee)_{-d+k}\\
&= \dim(R/N)_{d-k} - \dim(R)_{d-k+1}\\
&= h''_{d-k}(\Delta) - h'_{d-k+1}(\Delta)\\
&= h''_{d-k}(\Delta) - h''_{d-k+1}(\Delta) - \binom{d}{d-k+1}\beta_{d-k}(\Delta)\\
&= h''_{k}(\Delta) - h''_{k-1}(\Delta) - \binom{d}{k-1}\beta_{k-1}(\Delta)\\
&= h'_{k}(\Delta) - h''_{k-1}(\Delta) - \binom{d+1}{k}\beta_{k-1}(\Delta)\\
&= \widetilde{g}_k(\Delta).
\end{aligned}$$

Thus

$$\dim_{\mathbb{F}}(R'/J)_k = \dim_{\mathbb{F}}(\mathrm{Im}\phi/\theta_{d+1}R^\vee)_{-d+1} = \widetilde{g}_k(\Delta)$$

for $k \leq \frac{d}{2}$. Also, since $\mathrm{Im}\phi/\theta_{d+1}R^\vee$ is a quotient of $\mathrm{Im}\phi/\theta_{d+1}\mathrm{Im}\phi \cong (R/N)/\theta_{d+1}(R/N)$, the surjectivity in (II) tells $(R'/J)_k = 0$ for $k > \frac{d}{2}$, so $\mathrm{Hilb}(R'/J, t) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \widetilde{g}_k(\Delta)t^k$ as desired. □

Remark 4.3. For the proof, we follow the argument in [MN]. It would be possible to prove this theorem using the partition complex, a recent argument given in [AY], which may give a better understanding for the ideal J.

Theorem 3.2 easily follows from the previous result.

Proof of Theorem 3.2. Let Θ, R' and J be as in Theorem 4.2. Since $R' \cong S/(I_\Delta + (\Theta))$ and $S/(\Theta) \cong R = \mathbb{F}[x_1, \dots, x_{n-d-1}]$, we have $R'/J \cong R/I$ for some ideal I in R. It suffices to show that $\dim_{\mathbb{F}} I_2 \geq \binom{d+1}{2}\beta_1(\Delta)$. But this inequality follows from the trivial inequality $\dim_{\mathbb{F}} I_2 \geq \dim_{\mathbb{F}} J_2$ and Theorem 4.2(2-a). □

5. Example

We will not give a general construction used to prove Theorem 3.4, but present one example of a triangulation Γ of $\mathbb{S}^5 \times \mathbb{S}^1$ whose $\widetilde{g}$-vector coincides with the Hilbert series of $\mathbb{F}[x_1, \dots, x_8]/(x_1, \dots, x_7)^2$, that is, $\widetilde{g}(\Gamma) = (1, 8, 8, 8)$.

Consider the simplicial complex B on the vertex set $\{0,1,\dots,8,\bar{1},\dots,\bar{7},v_0,\dots,v_6\}$ generated by the following simplices

01234567	01234578	01235678	01345678
$\bar{7}1234567$	$\bar{7}1234578$	$\bar{7}1235678$	$\bar{7}1345678$
$\bar{6}\bar{7}234567$	$\bar{6}\bar{7}234578$	$\bar{6}\bar{7}235678$	
$\bar{5}\bar{7}\bar{6}34567$	$\bar{5}\bar{7}\bar{6}34578$	$\bar{5}\bar{7}\bar{6}35678$	$\bar{5}\bar{7}345678$
$\bar{4}\bar{7}\bar{6}\bar{5}4567$	$\bar{4}\bar{7}\bar{6}\bar{5}4578$		$\bar{4}\bar{7}\bar{5}45678$
$\bar{3}\bar{7}\bar{6}\bar{5}\bar{4}567$	$\bar{3}\bar{7}\bar{6}\bar{5}\bar{4}578$	$\bar{3}\bar{7}\bar{6}\bar{5}5678$	$\bar{3}\bar{7}\bar{5}\bar{4}5678$
$\bar{2}\bar{7}\bar{6}\bar{5}\bar{4}\bar{3}67$		$\bar{2}\bar{7}\bar{6}\bar{5}\bar{3}678$	$\bar{2}\bar{7}\bar{5}\bar{4}\bar{3}678$
$\bar{1}\bar{7}\bar{6}\bar{5}\bar{4}\bar{3}\bar{2}7$	$\bar{1}\bar{7}\bar{6}\bar{5}\bar{4}\bar{3}78$	$\bar{1}\bar{7}\bar{6}\bar{5}\bar{3}\bar{2}78$	$\bar{1}\bar{7}\bar{5}\bar{4}\bar{3}\bar{2}78$
$\bar{7}\bar{6}\bar{5}\bar{4}\bar{3}\bar{2}\bar{1}v_0$	$\bar{6}\bar{5}\bar{4}\bar{3}\bar{2}\bar{1}v_0v_1$	$\bar{5}\bar{4}\bar{3}\bar{2}\bar{1}v_0v_1v_2$	$\bar{4}\bar{3}\bar{2}\bar{1}v_0v_1v_2v_3$
$\bar{3}\bar{2}\bar{1}v_0v_1v_2v_3v_4$	$\bar{2}\bar{1}v_0v_1v_2v_3v_4v_5$	$\bar{1}v_0v_1v_2v_3v_4v_5v_6$	

The simplicial complex B is a shellable triangulated 7-dim ball with the h-vector

$$h(B)=(1,15,8,8,4,0,0,0,0)$$

and it boundary ∂B has the g-vector

$$g(\partial B)=(1,15,8,8).$$

Note that 0123456 and $v_0v_1v_2v_3v_4v_5v_6$ are facets of ∂B. Let Γ be the simplicial complex obtained from $\partial B\setminus\{0123456, v_0v_1v_2v_3v_4v_5v_6\}$ by identifying $v_0\to 0,\dots,v_6\to 6$. Note that the vertex set of Γ is $\{0,1,\dots,8,\bar{1},\dots,\bar{7}\}$. The operation $\partial B\to\Gamma$ is a combinatorial handle addition (see [NS1, §5]), so $|\Gamma|$ is homeomorphic to either $\mathbb{S}^5\times\mathbb{S}^1$ or $\mathbb{S}^5 \underline{\times}\, \mathbb{S}^1$. A routine computation tells that Γ is not orientable, so it must triangulates $\mathbb{S}^5 \underline{\times}\, \mathbb{S}^1$. Also, by a simple counting, we can see that

$$\widetilde{g}(\Gamma)=g(\partial B)-(0,7,0,0)=(1,8,8,8).$$

6. Related Problems

Here we add a few more problems relating to Problem 3.3.

Construction of the equality case. Theorem 3.4 is not perfect since it say nothing about the case $k=\frac{d}{2}$. It would be desirable to have a construction including $k=\frac{d}{2}$ case.

Problem 6.1. Let $d\geq 4$ be an even integer, $n\geq 2d+1$ and $R=\mathbb{R}[x_1,\dots,x_{n-d-1}]$. Is there an n vertex triangulation of $\mathbb{S}^{d-2}\times\mathbb{S}^1$ such that

$$\widetilde{g}_k(\Delta)=\dim_{\mathbb{F}}(R/(x_1,\dots,x_d)^2)_k \quad \text{for} \quad k=0,1,\dots,\tfrac{d}{2}?$$

More general upper bounds. In Problem 3.3, we consider $\mathbb{S}^{d-2}$-bundles over $\mathbb{S}^1$ since triangulations of these manifolds are well-studied, but of course a bound which holds for more general manifolds would be desirable. The next problem might be a natural generalization of Problem 3.3.

Problem 6.2. Let Δ be an n vertex triangulation of a connected closed $(d-1)$-dimensional manifold with $\beta_\ell(\Delta)\neq 0$, where $\ell<\frac{d}{2}$. Is it true that

$$\widetilde{g}_k(\Delta)\leq \dim_{\mathbb{F}}(\mathbb{F}[x_1,\dots,x_{n-d-1}]/(x_1,\dots,x_{d+1-\ell})^{\ell+1})_k \quad \text{for } k\leq \tfrac{d}{2}?$$

Interesting special cases of the above problem would be triangulations of the sphere product $\mathbb{S}^\ell \times \mathbb{S}^{d-1-\ell}$.

It would be also interesting to consider similar guess for manifolds M such that $\beta_k(M) \neq 0$ for many k like a projective space $\mathbb{RP}^d$ or a torus $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. But, at this moment, it is not clear how we can find a concrete guess on upper bounds of $\widetilde{g}$-vectors for triangulations of these manifolds from Theorem 4.2.

Buchsbaum complexes. The h''-vectors works well not only for triangulated manifold but also a more general class of simplicial complexes called Buchsbaum complexes. A $(d-1)$-dimensional simplicial complex Δ is said to be **Buchsbaum** (over $\mathbb{F}$) if $\widetilde{H}_k(\mathrm{lk}_\Delta(F);\mathbb{F}) \neq 0$ for $k \neq d-1-\#F$ for any non-empty face $F \in \Delta$. The following result is due to Novik and Swartz [NS1].

Theorem 6.3. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex with the vertex set $[n]$, Θ an l.s.o.p. for $\mathbb{F}[\Delta]$ and $R = \mathbb{F}[\Delta]/(\Theta\mathbb{F}[\Delta])$. There is a submodule $N \subset \mathrm{Soc}(R)$ such that*

1. $\mathrm{Hilb}(N,t) = \sum_{k=1}^{d-1} \binom{d}{k} \beta_{k-1}(\Delta) t^k$,
2. $\mathrm{Hilb}(R/N,t) = \sum_{k=0}^{d} h''_k(\Delta) t^k$.

Answers to the following problems may help understanding Problems 3.3, 6.1 and 6.2.

Problem 6.4. Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex having n vertices. Is it true that if $\beta_\ell(\Delta) \neq 0$ then

$$h''_k(\Delta) \leq \dim_{\mathbb{F}}(\mathbb{F}[x_1,\dots,x_{n-d}]/(x_1,\dots,x_{d-\ell})^{\ell+1})_k \quad \text{for } k = 0,1,\dots,d?$$

Problem 6.5. For all integers ℓ, d, n satisfying $0 \leq \ell \leq d-1$ and $n \geq 2d-\ell$, can we find a $(d-1)$-dimensional Buchsbaum simplicial complex Δ such that $\beta_\ell(\Delta) \neq 0$ and

$$h''_k(\Delta) = \dim_{\mathbb{F}}(\mathbb{F}[x_1,\dots,x_{n-d}]/(x_1,\dots,x_{d-\ell})^{\ell+1})_k \quad \text{for } k = 0,1,\dots,d?$$

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