INCREASING SUBSEQUENCES AND KRONECKER COEFFICIENTS

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A permutation π in the symmetric group $\mathfrak{S}_n$ is said to have an increasing subsequence of length k if there are numbers $1 \leq i_1 < \cdots < i_k \leq n$ such that

$$\pi(i_1) < \cdots < \pi(i_k).$$

Similarly, π has a decreasing subsequence of length l if we can find $1 \leq j_1 < \cdots < j_l \leq n$ with

$$\pi(j_1) > \cdots > \pi(j_l).$$

Increasing and decreasing subsequences in permutations have been the subject of sustained interest in combinatorics since 1935, when Erdős and Szekeres proved that, given any $k, l \in \mathbb{N}$, every permutation in $\mathfrak{S}_n$ contains either an increasing subsequence of length k or a decreasing subsequence of length l as soon as n exceeds $(k-1)(l-1)$. This is a permutation version of Ramsey's theorem in which the threshold between possible disorder and certain order is explicitly computable.

In view of the Erdős-Szekeres theorem, it is natural to wonder about the typical length of longest monotone subsequences in permutations. Since reversing a permutation interchanges its increasing and decreasing subsequences, we may focus on just one of the two orientations. Let LIS_n denote the length of the longest increasing subsequence in a uniformly random sample from $\mathfrak{S}_n$. What can be said about the distribution of LIS_n? This question has been answered in the $n \to \infty$ limit: we have both a Law of Large Numbers and a Central Limit Theorem for LIS_n. The LLN was was obtained by Vershik and Kerov in 1977, who showed that

$$\lim_{n\to\infty} \frac{\mathrm{LIS}_n}{\sqrt{n}} = 2,$$

the convergence being in probability. The corresponding CLT is a breakthrough 1999 result of Baik, Deift, and Johansson, who discovered that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathrm{LIS}_n - 2\sqrt{n}}{n^{1/6}} \leq t\right) = F(t),$$

where $F(t)$ is the Tracy-Widom distribution from random matrix theory. For an informative discussion of these results, together with pertinent references, we refer the reader to Stanley's 2006 ICM contribution [7].

The distribution of LIS_n at finite n remains poorly understood — even the question of unimodality has not been settled. Given that the distribution of most natural combinatorial statistics is unimodal, it is not unreasonable to conjecture that this holds true for LIS_n. The above limit theorems certainly suggest that this is the case, since they tell us that when n is large the distribution of LIS_n is concentrated in a small neighborhood of $2\sqrt{n}$. However, this does not rule out the possibility of irregular "noisy" behavior before the limit. In combinatorics and elsewhere, the source of unimodality is often a stronger property: log-concavity [6]. Conjecturally, this is the case for LIS_n.

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Conjecture 1 (Chen [2]). *For any $n \geq 3$, the distribution of LIS_n is log-concave. That is, we have*

$$a_{n,k-1}a_{n,k+1} \leq a_{n,k}^2, \tag{1}$$

for all $2 \leq k \leq n-1$, where $a_{n,k}$ is the number of permutations in $\mathfrak{S}_n$ whose longest increasing subsequence has length k.

The inequality (1) is equivalent to the existence of an injection

$$\mathfrak{S}_{n,k-1} \times \mathfrak{S}_{n,k+1} \longrightarrow \mathfrak{S}_{n,k} \times \mathfrak{S}_{n,k}, \tag{2}$$

where $\mathfrak{S}_{n,k}$ is the set of permutations in $\mathfrak{S}_n$ whose longest increasing subsequence has length equal to k. It seems difficult to construct such an injection in any uniform way: tampering with a permutation alters its increasing subsequences in complicated ways which are difficult to track. Instead, one may invoke the Robinson-Schensted correspondence [5], which tells us that $a_{n,k}$ is equal to the number of pairs of standard Young tableaux on a common shape $\lambda \vdash n$ satisfying $\ell(\lambda) = k$. Thus (2) is equivalent to the existence of an injection

$$\mathfrak{P}_{n,k-1} \times \mathfrak{P}_{n,k+1} \longrightarrow \mathfrak{P}_{n,k} \times \mathfrak{P}_{n,k}, \tag{3}$$

where $\mathfrak{P}_{n,k}$ is the set of Robinson-Schensted pairs with n cells and k rows. Since operations on Young tableaux are easier to visualize than operations on permutations, such an injection may be easier to construct. Another possibility would be to use the hook-length formula $f^\lambda = n!/H_\lambda$ for the number f^λ of SYT of shape λ, which combined with (3) gives

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k-1}} \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k+1}} \frac{1}{H_\lambda^2 H_\mu^2} \leq \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} \frac{1}{H_\lambda^2 H_\mu^2}. \tag{4}$$

This inequality is equivalent to (1), but may be easier to work with. So far, neither of these strategies has been successfully implemented; see [1] for some partial results.

When faced with an intractable problem, there is nothing to lose and everything to gain by trying to solve an even harder problem. We submit that the "right" way to approach Conjecture 1 is to focus on the following stronger conjecture. Let $R(\mathfrak{S}_n)$ be the representation ring of $\mathfrak{S}_n$, i.e. the commutative ring generated by the isomorphism classes V^λ of irreducible complex representations of $\mathfrak{S}_n$ with operations being direct sum and tensor product of $\mathfrak{S}_n$-modules. Let us replace the sets $\mathfrak{S}_{n,k}$ appearing in (2) with the modules $V_{n,k} \in R(\mathfrak{S}_n)$ defined by

$$V_{n,k} = \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} f^\lambda V^\lambda.$$

Since $\dim V^\lambda = f^\lambda$, we have

$$\dim V_{n,k} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} (f^\lambda)^2 = a_{n,k},$$

whence inequality (1) is equivalent to

$$\dim V_{n,k-1} \dim V_{n,k+1} \leq \dim V_{n,k} \dim V_{n,k}. \tag{5}$$

We propose the following strengthening of Conjecture 1.

Conjecture 2. *For any $n \geq 3$, there exists an $\mathfrak{S}_n$-equivariant injection*

$$V_{n,k-1} \otimes V_{n,k+1} \longrightarrow V_{n,k} \otimes V_{n,k}. \tag{6}$$

for all $2 \leq k \leq n-1$.

Conjecture 2 may equivalently be formulated as a numerical refinement of (4). To see this, we make use of the Kronecker coefficients, which linearize multiplication in $R(\mathfrak{S}_n)$:

$$V^\lambda \otimes V^\mu = \bigoplus_{\nu \vdash n} g^\nu_{\lambda\mu} V^\nu.$$

Decomposing the source and target in (6) into irreducibles, Conjecture 2 claims the existence of an injective $\mathfrak{S}_n$-module homomorphism

$$\bigoplus_{\nu \vdash n} \left(\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k-1}} \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k+1}} f^\lambda f^\mu g^\nu_{\lambda\mu} \right) V^\nu \longrightarrow \bigoplus_{\nu \vdash n} \left(\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} f^\lambda f^\mu g^\nu_{\lambda\mu} \right) V^\nu.$$

Thus, by Schur's lemma and the hook-length formula, Conjecture 2 is equivalent to the following numerical inequality.

Conjecture 3. *For any $n \geq 3$ and $\nu \vdash n$, we have*

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k-1}} \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k+1}} \frac{g^\nu_{\lambda\mu}}{H_\lambda H_\mu} \leq \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} \sum_{\substack{\mu \vdash n \\ \ell(\mu)=k}} \frac{g^\nu_{\lambda\mu}}{H_\lambda H_\mu} \tag{7}$$

for all $2 \leq k \leq n-1$.

The inequality (4) is recovered from Conjecture 3 by summing (7) over all $\nu \vdash n$.

Yet another equivalent formulation of Conjecture 2 may be obtained by means of the Frobenius isomorphism

$$\mathrm{Frob}\colon R(\mathfrak{S}_n) \longrightarrow \Lambda_n,$$

where Λ_n is the ring of homogeneous symmetric functions of degree n equipped with the Kronecker product. We recall that the Kronecker product in Λ_n is defined via bilinear extension of the rule

$$s_\lambda * s_\mu := \sum_{\nu \vdash n} g^\nu_{\lambda\mu} s_\nu, \tag{8}$$

where $\{s_\lambda \colon \lambda \vdash n\}$ is the Schur function basis. The Frobenius isomorphism is defined by $\mathrm{Frob}(V^\lambda) = s_\lambda$. In particular, the Frobenius image of the increasing subsequence module $V_{n,k} \in R(\mathfrak{S}_n)$ appearing in Conjecture 2 is the symmetric function $S_{n,k} \in \Lambda_n$ given by

$$S_{n,k} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=k}} f^\lambda s_\lambda.$$

Given symmetric functions $F, G \in \Lambda_n$, let us write $F \leq G$ if the difference $G - F$ is Schur positive, i.e. if the coefficients c_λ defined by

$$G - F = \sum_{\lambda \vdash n} c_\lambda s_\lambda$$

are nonnegative. Conjecture 2 may then be restated as follows.

Conjecture 4. *For any $n \geq 3$, we have*

$$S_{n,k-1} * S_{n,k+1} \leq S_{n,k} * S_{n,k}$$

for all $2 \leq k \leq n-1$.

Conjecture 4 is a useful equivalent formulation of Conjecture 2 in that Schur positivity is a well-developed topic in algebraic combinatorics. Conjecture 4 has been verified for $n \leq 15$ on a computer.

How could one go about proving Conjecture 2? Let us illustrate how a successful argument might look by outlining a representation-theoretic proof of a much simpler proposition: the log-concavity of binomial coefficients. From the hook-length formula, it is clear that the dimension of the irreducible representation of $\mathfrak{S}_n$ corresponding to the hook $\lambda = (n-k, 1^k)$ is given by $\binom{n-1}{k}$ for all $0 \leq k \leq n-1$:

$$\dim V^{(n-k,1^k)} = \binom{n-1}{k}. \tag{9}$$

The difference $\binom{n-1}{k}^2 - \binom{n-1}{k-1} \cdot \binom{n-1}{k+1}$ may therefore be expressed as a difference of dimensions of Kronecker products:

$$\binom{n-1}{k}^2 - \binom{n-1}{k-1} \cdot \binom{n-1}{k+1} = \dim(V^{(n-k,1^k)} \otimes V^{(k+1,1^{n-k-1})}) - \dim(V^{(n-k+1,1^{k-1})} \otimes V^{(k+2,1^{n-k-2})}). \tag{10}$$

The sequence of binomial coefficients with upper index $n-1$ will thus be certified log-concave if we can exhibit an $\mathfrak{S}_n$-module $V_{n,k}$ whose Frobenius image is

$$\mathrm{Frob}(V_{n,k}) = s_{(n-k,1^k)} * s_{(k+1,1^{n-k-1})} - s_{(n-k+1,1^{k-1})} * s_{(k+2,1^{n-k-2})}. \tag{11}$$

The required module was found by Kim and Rhoades [4]. Let $\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n$ be a list of $2n$ anticommuting variables and consider the exterior algebra

$$\wedge\{\Theta_n, \Xi_n\} := \wedge\{\theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n\} \tag{12}$$

over $\mathbb{C}$ generated by these variables. This is a $\mathbb{C}$-vector space of dimension 2^{2n} which carries the bigrading

$$\wedge\{\Theta_n, \Xi_n\} = \bigoplus_{i,j=0}^{n} \wedge\{\Theta_n, \Xi_n\}_{i,j} \tag{13}$$

induced by considering the degree of the θ-variables and ξ-variables separately. In Physics, anticommuting variables are called "fermionic," and the relation $\theta_i^2 = 0$ corresponds to the Pauli Exclusion Principle: no two fermions may occupy the same state at the same time. Consider the diagonal action of $\mathfrak{S}_n$ on $\wedge\{\Theta_n, \Xi_n\}$, viz.

$$w \cdot \theta_i := \theta_{w(i)} \qquad w \cdot \xi_i := \xi_{w(i)} \qquad w \in \mathfrak{S}_n, \ 1 \leq i \leq n \tag{14}$$

and denote by $\langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n} \rangle \subseteq \wedge\{\Theta_n, \Xi_n\}$ the two-sided ideal generated by $\mathfrak{S}_n$-invariants with vanishing constant term. The *fermionic diagonal coinvariant* ring is defined in [4] by

$$FDR_n := \wedge\{\Theta_n, \Xi_n\} / \langle \wedge\{\Theta_n, \Xi_n\}_+^{\mathfrak{S}_n} \rangle. \tag{15}$$

This is a doubly graded $\mathfrak{S}_n$-module and an anticommutative version of the *diagonal coinvariant ring* [3].

Theorem 5. *(Kim-R. [4]) The (i,j)-graded piece $(FDR_n)_{i,j}$ is zero unless $i + j < n$. When $i + j < n$ we have*

$$\mathrm{Frob}((FDR_n)_{i,j}) = s_{(n-i,1^i)} * s_{(n-j,1^j)} - s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})}, \tag{16}$$

where we interpret $s_{(n-i+1,1^{i-1})} * s_{(n-j+1,1^{j-1})} = 0$ *if* $i = 0$ *or* $j = 0$*. In particular, for* $0 \leq k \leq n-1$ *if we set* $V_{n,k} := (FDR_n)_{k,n-k-1}$ *then* $V_{n,k}$ *has Frobenius image given by Equation* (11)*.*

Theorem 5 implies that the symmetric function in Equation (11) is Schur-positive, and taking vector space dimensions yields the log-concavity of the sequence $\binom{n-1}{0}, \binom{n-1}{1}, \ldots, \binom{n-1}{n-1}$.

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