Positive Structures in Algebraic Combinatorics

Sami H. Assaf

University of Southern California

Open Problems in Algebraic Combinatorics
May 2022

Outline

1. Schur positivity
   - Characters, Setting, and Plot
   - Point of View
   - Problems and Resolutions

2. Schur P-positivity
   - Shifty Characters
   - Mind your P's and Q's
   - Words, words, words

3. Demazure positivity
   - Trimming characters
   - Excellent examples
   - Setting up sequels

Outline

1. Schur positivity
   - Characters, Setting, and Plot
   - Point of View
   - Problems and Resolutions
2. Schur P-positivity
   - Shifty Characters
   - Mind your P's and Q's
   - Words, words, words
3. Demazure positivity
   - Trimming characters
   - Excellent examples
   - Setting up sequels

General Linear group | Symmetric Group

Schur modules $\{V^\lambda \mid \ell(\lambda) \leq n\}$ are the irreducible representations of GL_n, and

$$\mathrm{char}(V^\lambda) = s_\lambda(x_1, \ldots, x_n)$$

$\begin{smallmatrix}2 & \\ 1 & 1\end{smallmatrix}$ $(2,1,0)$	$\begin{smallmatrix}2 & \\ 1 & 2\end{smallmatrix}$ $(1,2,0)$	$\begin{smallmatrix}3 & \\ 1 & 1\end{smallmatrix}$ $(2,0,1)$	$\begin{smallmatrix}2 & \\ 1 & 3\end{smallmatrix}$ $(1,1,1)$
$\begin{smallmatrix}3 & \\ 1 & 2\end{smallmatrix}$ $(1,1,1)$	$\begin{smallmatrix}3 & \\ 1 & 3\end{smallmatrix}$ $(1,0,2)$	$\begin{smallmatrix}3 & \\ 2 & 2\end{smallmatrix}$ $(0,2,1)$	$\begin{smallmatrix}3 & \\ 2 & 3\end{smallmatrix}$ $(0,1,2)$

$$s_{(2,1,0)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 + x_1 x_3^2 + x_2 x_3^2$$

Let $x^{\mathbf{a}} = x_1^{\mathbf{a}_1} \cdots x_n^{\mathbf{a}_n}$ be the monomial basis for polynomials.

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda)} x^{\mathbf{wt}(T)}$$

Specht modules $\{S^\lambda \mid |\lambda| = n\}$ are the irreducible representations of $\mathcal{S}_n$, and

$$\mathrm{Frob}(S^\lambda) = s_\lambda(x_1, x_2, \ldots)$$

$\begin{smallmatrix}2 & & \\ 1 & 3 & 4\end{smallmatrix}$ $(1,3)$	$\begin{smallmatrix}3 & & \\ 1 & 2 & 4\end{smallmatrix}$ $(2,2)$	$\begin{smallmatrix}4 & & \\ 1 & 2 & 3\end{smallmatrix}$ $(3,1)$
$\begin{smallmatrix}2 & 4 \\ 1 & 3\end{smallmatrix}$ $(1,2,1)$	$\begin{smallmatrix}3 & 4 \\ 1 & 2\end{smallmatrix}$ $(2,2)$	

$$s_{(3,1)} = F_{(1,3)} + F_{(2,2)} + F_{(3,1)}$$
$$s_{(2,2)} = F_{(1,2,1)} + F_{(2,2)}$$

Let $F_\alpha = \sum_{\mathbf{a} \text{ refines } \alpha} x_1^{\mathbf{a}_1} \cdots x_n^{\mathbf{a}_n}$ be the fundamental basis for quasisymmetric functions.

$$s_\lambda(x_1, x_2, \ldots) = \sum_{S \in \mathrm{SYT}(\lambda)} F_{\mathrm{Des}(S)}$$

Crystal graphs

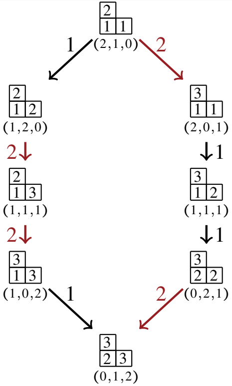
Dual equivalence graphs

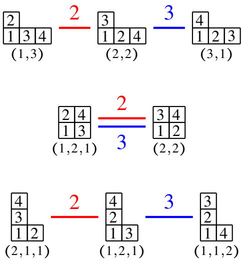
Theorem (Kashiwara 1991, Kashiwara–Nakashima 94, Littelmann 95)

Explicit crystal pairing rule for $i, i+1$, and f_i changes the first unpaired i to $i+1$.

Theorem (Knuth 1979, Edelman–Greene 1987, Haiman 1992, A. 2007)

Explicit witness rule for $i-1, i, i+1$ and d_i swaps i with the non-witness of $i \pm 1$.

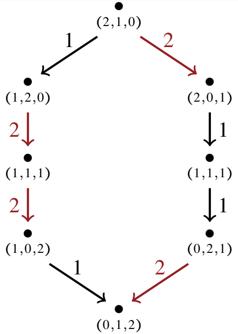




Crystal axioms

Theorem (Stembridge 2003)

If a graph is locally a crystal, then it is globally a crystal.

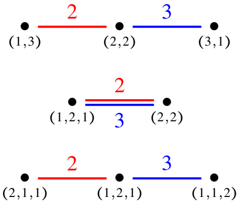


Every connected crystal $B(\lambda)$ has a unique highest weight element U, characterized by $e_i(U) = 0$ for all i, and $\mathbf{wt}(U) = \lambda$.

Dual equivalence axioms

Theorem (A. 2007/15)

If a graph is locally a dual equivalence, then it is globally a dual equivalence.



Open Problem

Find an analog of highest weights for dual equivalence to facilitate “nice” formulas.

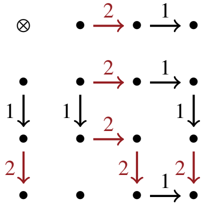
Littlewood–Richardson rule gives the coefficients in tensor products: $V^{\mu} \otimes V^{\nu} = \bigoplus_{\lambda} (V^{\lambda})^{\oplus c^{\lambda}_{\mu,\nu}}$

$$s_{\mu} s_{\nu} = \sum_{\lambda} c^{\lambda}_{\mu,\nu} s_{\lambda}$$

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu}$$

Define the tensor edges $f_i(b_1 \otimes b_2)$ by

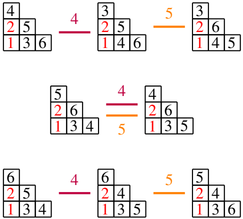
$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2 & \text{if } \mathrm{head}_i(b_2) < \mathrm{tail}_i(b_1), \\ b_1 \otimes f_i(b_2) & \text{if } \mathrm{head}_i(b_2) \geq \mathrm{tail}_i(b_1). \end{cases}$$



$$s_{(1,1,0)} s_{(1,0,0)} = s_{(2,1,0)} + s_{(1,1,1)}$$

$c^{\lambda}_{\mu,\nu}$ = number of lattice elements

Define the skew of an equivalence by ignoring $d_2, \ldots, d_{k+1}$ for $1, \ldots, k$ in fixed positions.



$$s_{(3,2,1)/(1,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}$$

$c^{\lambda}_{\mu,\nu}$ = number of elements that rectify to S_{λ}

Stanley symmetric functions

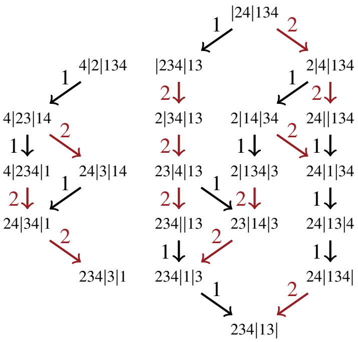
Stanley (1984) defined a symmetric function $S_w = \sum_{\rho \in \mathrm{Red}(w)} F_{\mathrm{Des}(\rho)}$ over reduced words for w.

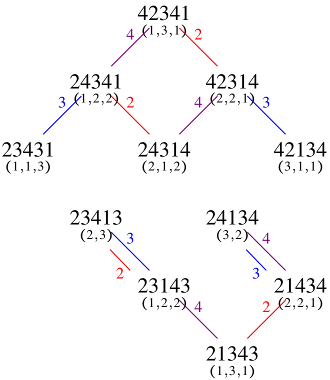
Theorem (Morse–Schilling 2016)

Explicit crystal directly on increasing factorizations of reduced words for w.

Theorem (Edelman–Greene 1987)

Coxeter–Knuth relations are a dual equivalence on reduced words for w.





Macdonald polynomials

Macdonald polynomials (1988) are expressed by Haglund–Haiman–Loehr (2005) as:

$$\widetilde{H}_\mu(x;q,t) = \sum_{w \in \mathcal{S}_n} q^{\mathrm{inv}(w)} t^{\mathrm{maj}(w)} F_{\mathrm{Des}(w)}$$

Theorem (Hall–Littlewood case)

There is a crystal for $\widetilde{H}_\mu(x;0,t)$ for which maj is the energy function.

Theorem (van Leeuwen 2005)

There is a crystal for $\widetilde{H}_\mu(x;q,t)$ for $\mu_1 \leq 2$, where maj and inv are constant on connected components.

Open Problem

Find a crystal for $\widetilde{H}_\mu(x;q,t)$ (or $\mathrm{LLT}_{\boldsymbol{\mu}}(x;q)$).

Possibly this can be done with a q-analog of the tensor rule corresponding to the q-product for LLT polynomials.

Theorem (A. 2007/15)

There is a dual equivalence for $\widetilde{H}_\mu(x;q,t)$ for $\mu_1 \leq 2$, where both maj and inv are constant on classes.

Theorem (A. 2007/15)

Both maj and inv are constant on twisted dual equivalence classes. Moreover, for $\mu_1 \leq 2$ and $\mu = (n)$ these are Schur positive.

Theorem (Roberts 2013)

Twisted classes are positive for $\mathrm{inv} = 0$.

Conjecture (A. 2015)

Twisted classes are Schur positive.

Chromatic symmetric functions

Stanley (1995) defined a symmetric function $X_G = \sum_{\kappa:V(G)\to\mathbb{N}} x^{\mathbf{wt}(\kappa)}$ generalizing the chromatic number of a graph G. Stanley conjectured $X_{\mathrm{inc}(\mathcal{P})}$ is Schur positive for certain $\mathcal{P}$.

Theorem (Gasharov 1995)

For $G = \mathrm{inc}(\mathcal{P})$ with $\mathcal{P}$ a $(3+1)$-free poset, $X_G(x) = \sum_T s_{\mathbf{wt}(T)}$ where the sum is over $\mathcal{P}$-tableaux.

Theorem (Ehrhard 2022+)

There is a crystal structure refining the Schur positivity, and for 2-part Schur polynomials, this is a Stembridge crystal. Moreover, the Shareshian–Wachs statistic factors through.

Open Problem

Complete this result to the general case.

Theorem (Kim–Pylyavskyy 2021)

With additional (conjecturally unnecessary) conditions on $\mathcal{P}$, there is an insertion algorithm proved using dual equivalence for proper colorings of $\mathrm{inc}(\mathcal{P})$.

Open Problem

Complete this result to the general $(3+1)$-free case.

Outline

Queer superalgebras — Projective representations

Consider strict partitions $\gamma = (\gamma_1 > \gamma_2 > \cdots > \gamma_\ell)$ indexing a subspace of symmetric functions. Schur P- and Q-functions (given by $P_\gamma(x;-1)$ and $Q_\gamma(x;-1)$, resp.) are dual bases.

Theorem (Sergeev 1984)

Schur P-polys are characters of tensor representations of queer superalgebras.

Theorem (Stembridge 1989)

Schur P-functions are characters of projective representations of S_n.

$$P_\gamma(x_1,\ldots,x_n) = \sum_{T \in \mathrm{SSHT}_n(\gamma)} x^{\mathbf{wt}(T)}$$

2 / 1 1	(2,1,0)
2 / 1 2′	(1,2,0)
3 / 1 1	(2,0,1)
3 / 1 2	(1,1,1)
3 / 1 2′	(1,1,1)
3 / 1 3′	(1,0,2)
3 / 2 2	(0,2,1)
3 / 2 3′	(0,1,2)

$$P_{(2,1,0)} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_2x_3 + x_1x_2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2$$

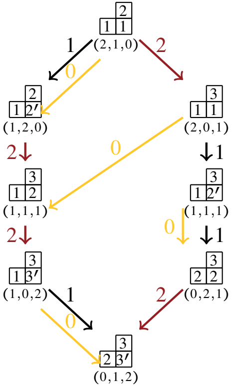
$$P_\gamma(x_1,x_2,\cdots) = \sum_{S \in \mathrm{SHT}(\gamma)} F_{\mathrm{Des}(S)}$$

3 / 1 2′ 4	(1,3)
3 / 1 2 4	(2,2)
4 / 1 2 3	(3,1)
4 / 1 2′ 3	(1,2,1)
4 / 1 2 3′	(2,2)
4 / 1 2′ 3′	(1,1,2)
3 / 1 2′ 4′	(1,2,1)
3 / 1 2 4′	(2,1,1)

$$P_{(3,1)} = F_{(1,3)} + F_{(2,2)} + F_{(3,1)} + F_{(1,2,1)} + F_{(2,2)} + F_{(1,1,2)} + F_{(1,2,1)} + F_{(2,1,1)}$$

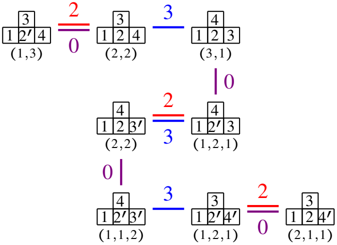
Theorem (GJKKK 2010, A.–Oguz 2018)

Explicit crystal pairing rule for $i, i+1$*, and*
f_i *changes the first unpaired* i *to* $i+1$*,*
f_0 *changes the rightmost* 1 *in bottom row.*



Theorem (A. 2022)

Explicit witness rule for $i-1, i, i+1$ *and*
d_i *swaps* i *with the non-witness of* $i \pm 1$*,*
d_0 *toggles the sign on the* 2*.*



Theorem

Erasing the queer edges recovers the type A structure for both paradigms. In particular, P_γ *is Schur positive.*

Queer crystal axioms

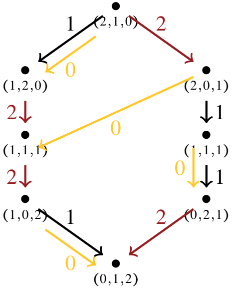
Queer dual equivalence axioms

Theorem (A.-Oguz 2020)

Short list of necessary local axioms using odd Kashiwara operators.

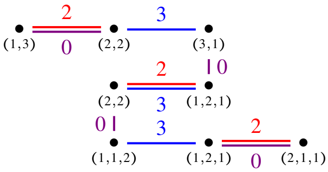
Theorem (Gillespie-Hawkes-Poh-Schilling)

Additional necessary local axioms, and sufficient condition on the Schur expansion.



Theorem (A. 2022)

Queer dual equivalence is characterized by 5 local axioms and 1 non-local axiom.



Open Problem

Give simple, local characterization of highest weights for queer crystals.

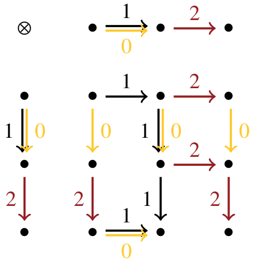
Open Problem

Give fully local axioms for queer dual equivalence (maybe using *odd* edges?).

Recall P_γ and Q_γ are dual bases, and so $P_\delta P_\epsilon = \sum_\gamma c^\gamma_{\delta,\epsilon} P_\gamma$ if and only if $Q_{\gamma/\delta} = \sum_\epsilon c^\gamma_{\delta,\epsilon} Q_\epsilon$.

Add the queer tensor edge $f_0(b_1 \otimes b_2)$ by

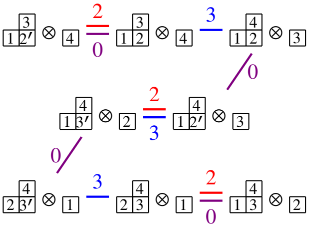
$$f_0(b_1 \otimes b_2) = \begin{cases} f_0(b_1) \otimes b_2 & \text{if } \mathbf{wt}(b_2)_1 = \mathbf{wt}(b_2)_2 = 0, \\ b_1 \otimes f_0(b_2) & \text{otherwise.} \end{cases}$$



$$P_{(1,0,0)} P_{(1,0,0)} = P_{(2,0,0)}$$

Define skew equivalence as before, but on the shifted dual equivalence (Billey–Hamacker–Roberts–Young 2014, A. 2018).

Or rely on the axioms to guide the rule.



$$P_{(2,1)} P_{(1)} = P_{(3,1)}$$

Reduced words

Billey–Haiman (1995) define the type C Stanley symmetric function signed permutations:

$$C_w(x) = \sum_{\rho \in \mathrm{Red}(w)} \Theta_{\mathrm{Peak}(\rho)}(x)$$

$$\mathrm{Red}(31'2') = \{(1,0,1,2,0,1),\ (1,0,1,0,2,1),\ (0,1,0,1,2,1),\ (0,1,0,2,1,2),\ (0,1,2,0,1,2)\}$$

Theorem (Hawkes–Paramonov–Schilling 2017)

Type C crystal on factorizations of reduced words signed permutations.

Theorem (Billey–Hamacker–Roberts–Young 2014)

Coxeter–Knuth relations give a shifted dual equivalence on these reduced words.

The type B Stanley symmetric function B_w is over signed reduced words where 0 is unsigned:

$$B_w(x) = \sum_{\rho \in \mathrm{Red}_0^{\pm}(w)} F_{\mathrm{Des}(w)}(x)$$

Open Problems

Define a queer crystal on factorizations of signed reduced words signed permutations.

Open Problems

Define a queer dual equivalence for signed reduced words signed permutations.

Other interesting words for which there are queer crystals and queer dual equivalence: involution Stanley symmetric functions, fixed-point-free Stanley symmetric functions, etc

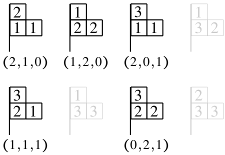
Outline

Demazure characters — Slide polynomials

Demazure modules $\{V_w^\lambda \mid w \in \mathcal{S}_n\}$ are B-submodules generated by extremal weight spaces,

$$\kappa_{\mathbf{a}}(x_1, \ldots, x_n) = \mathrm{char}(V_w^\lambda) = \pi_w(x^\lambda)$$

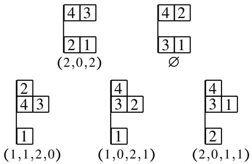
Kohnert diagrams (Kohnert 1991), Semi-skyline augmented fillings (Mason 2009)



$$\kappa_{(0,2,1)} = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3$$

$$\kappa_{\mathbf{a}} = \sum_{T \in \mathrm{SSAF}(\mathbf{a})} x^{\mathbf{wt}(T)}$$

Quasi-Yamanouchi Kohnert tableau (A.–Searles 2018), key tableaux (A. 2018)



$$\kappa_{(2,0,2)} = \mathfrak{F}_{(2,0,2)} + 0$$

$$\kappa_{(1,0,2,1)} = \mathfrak{F}_{(1,1,2,0)} + \mathfrak{F}_{(1,0,2,1)} + \mathfrak{F}_{(2,0,1,1)}$$

$$\kappa_{\mathbf{a}} = \sum_{S \in \mathrm{SKT}(\mathbf{a})} \mathfrak{F}_{\mathrm{des}(S)}$$

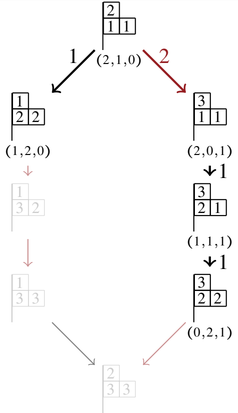
Here $\mathfrak{F}_{\mathbf{b}} = \sum_{\mathbf{a} \text{ refines } \mathbf{b}} x^a$ is the fundamental slide basis of the polynomial ring (A.-Searles 2017).

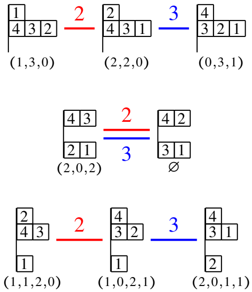
Theorem (A.-Schilling 2018)

Explicit crystal pairing rule for $i, i+1$*, and* f_i *changes the first unpaired* i *to* $i+1$*, and maybe flips* $i, i+1$ *in some columns.*

Theorem (A. 2022)

Explicit witness rule for $i-1, i, i+1$ *and* d_i *swaps* i *with the non-witness of* $i \pm 1$*, or maybe cyclically rotates* $i-1, i, i+1$.





Demazure axioms

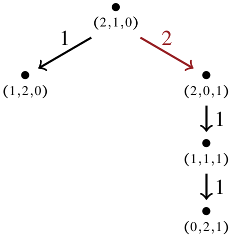
Weak dual equivalence axioms

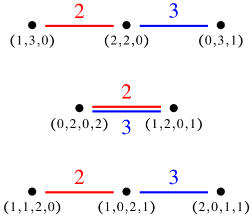
Theorem (A.-Gonzalez 2021)

If an extremal subgraph of a crystal satisfies two additional local axioms, then it is a Demazure crystal.

Theorem (A. 2022)

If the graph is a dual equivalence and locally Demazure positive (no virtual elements!), then it is a weak dual equivalence.





Open Problems

- Relate these sets of axioms, similar to the duality between crystals and dual equivalence.
- Replicate both sets of axioms explicitly for the queer/shifted case (Demazure crystals exist!).

Excellent filtrations

Theorem (Mathieu 1989)

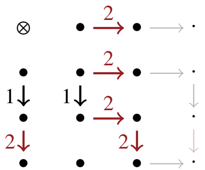
$V_u^\mu \otimes V^\nu$ has an excellent filtration $F_0 \subseteq F_1 \subseteq \cdots$ with $\bigcup_i F_i = V_u^\mu \otimes V^\nu$ s.t. $F_{i+1}/F_i \cong \bigoplus_j (V_{w_j}^{\lambda_j})^{\oplus m_j}$

Theorem (Haglund–Luoto–Mason–van Willigenberg 2011)

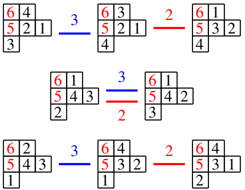
$$\mathrm{SSAF}_n(\mathbf{a}) \times \mathrm{SSYT}_n(\nu) \xrightarrow{\mathrm{RSK}} \bigsqcup_{\mathbf{c}} \mathrm{SSAF}_n(\mathbf{c})^{\oplus c^{\mathbf{c}}_{\mathbf{a},\nu}}$$

Theorem (A. 2022)

Ignoring $d_k, \ldots, d_{n-1}$ for $k+1, \ldots, n$ in fixed positions is a weak dual equivalence on $\mathbf{a}/\nu$.



$$\kappa_{(1,0,1)}\kappa_{(0,0,1)} = \kappa_{(1,0,2)} + \kappa_{(1,1,1)}$$



$$\kappa_{(1,3,2)/(1,1)} = \kappa_{(3,0,1)} + \kappa_{(2,2,0)} + \kappa_{(1,2,1)}$$

Open Problem

How are the coefficients in $\kappa_{\mathbf{a}} s_\nu = \sum_{\mathbf{c}} c^{\mathbf{c}}_{\mathbf{a},\nu} \kappa_{\mathbf{c}}$ and $\kappa_{\mathbf{c}/\nu} = \sum_{\mathbf{c}} \tilde{c}^{\mathbf{c}}_{\mathbf{a},\nu} \kappa_{\mathbf{c}}$ related? Are they equal?

When excellence is unattainable

$$\kappa_{(1,0,1)}\kappa_{(0,1,0)} = \kappa_{(1,1,1)} + \kappa_{(2,0,1)} + \kappa_{(1,2,0)} - \kappa_{(2,1,0)}$$

Conjecture (Polo 1989)

$V_u^\mu \otimes V_v^\nu$ has a Schubert filtration: $F_0 \subseteq F_1 \subseteq \cdots$ with $\bigcup_i F_i = V_u^\mu \otimes V_v^\nu$ s.t. $F_{i+1}/F_i \cong \bigoplus_j (U_{w_j}^{\lambda_j})^{\oplus m_j}$

Theorem (A.–Quijada 2019+, A. 2021+)

We have a bijection $\mathrm{KD}(\mathbf{a}) \times \mathrm{SSYT}_k(\nu) \xrightarrow{\sim} \bigcup_{\mathbf{c}} \mathrm{KD}(\mathbf{c})^{\oplus c_{\mathbf{a},\nu}^{\mathbf{c}}}$*, with* $c_{\mathbf{a},\nu}^{\mathbf{c}}$ *explicit, nonnegative.*

$$\mathrm{KD}(1,0,1) \times \mathrm{SSYT}_2(1) = \mathrm{KD}(1,1,1) \cup \mathrm{KD}(2,0,1) \cup \mathrm{KD}(1,2,0)$$

Almost Closed Problem

Give a crystal model of this positivity.

Open Problem

Give a dual equivalence for this positivity.

Open Problem

Give a bijection $\mathrm{KD}(\mathbf{a}) \times \mathrm{KD}(\mathbf{b}) \xrightarrow{\sim} \bigcup_{\mathbf{c}} \mathrm{KD}(\mathbf{c})^{\oplus c_{\mathbf{a},\mathbf{b}}^{\mathbf{c}}}$ with $c_{\mathbf{a},\mathbf{b}}^{\mathbf{c}}$ explicit, nonnegative integers.

Nonsymmetric Macdonald polynomials

Opdam's nonsymmetric Macdonald polynomials (1995) are expressed by Haglund–Haiman–Loehr (2008) as:

$$E_{\mathbf{a}}(x;q,t) = \sum_{\substack{T:a\to[n] \\ \text{non-attacking}}} q^{\mathrm{maj}(T)} t^{\mathrm{coinv}(T)} x^{\mathbf{wt}(T)} \prod_{c\neq \mathrm{left}(c)} \frac{1-t}{1-q^{\mathrm{leg}(c)+1} t^{\mathrm{arm}(c)+1}}$$

Theorem (A.–Gonzalez 2021)

Demazure crystal on semistandard key tabloids that preserves maj*, and so* $E_{\mathbf{a}}(x;q,0) = \sum_T q^{\mathrm{maj}(T)} \kappa_{\mathbf{wt}(T)}$.

Theorem (A. 2018)

$$E_{\mathbf{a}}(x;q,0) = \sum_{\substack{S:a\xrightarrow{\sim}[n] \\ \mathrm{coinv}(S)=0}} q^{\mathrm{maj}(S)} \mathfrak{F}_{\mathrm{des}(S)}$$

Theorem (A.–Gonzalez 2021)

Affine Demazure crystal on semistandard key tabloids with energy function maj*. Thus affine Demazure modules have excellent filtrations into finite Demazure modules.*

Theorem (A. 2018)

There is a weak dual equivalence on standard key tabloids that preserves maj*, and so* $E_{\mathbf{a}}(x;q,0)$ *is a nonnegative sum of Demazure characters.*

Open Problem

Prove affine Demazure modules have (finite) excellent filtrations in all types.

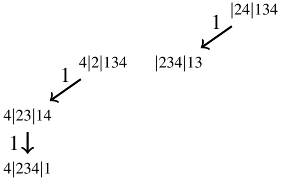
Schubert polynomials

Billey–Jockush–Stanley (1993), A.–Searles (2017), express Schubert polynomials as

$$\mathfrak{S}_w = \sum_{\rho \in \mathrm{Red}(w)} \left(\alpha \sum_{\rho-\text{compatible}} x^{\mathbf{wt}(\rho)} \right) = \sum_{\rho \in \mathrm{Red}(w)} \mathfrak{F}_{\mathrm{des}(\rho)}$$

Theorem (A.–Schilling 2016)

The Morse–Schilling crystal for S_w truncates to a Demazure crystal for $\mathfrak{S}_w$.

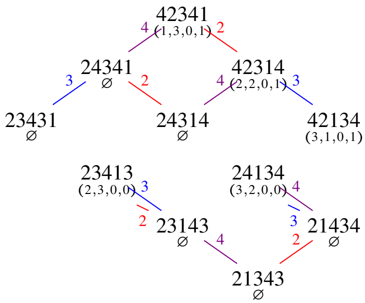


Open Problems

Truncate other Stanley functions to Schubert polynomials: type B, type C, involution, FPF.

Theorem (A. 2022)

Coxeter–Knuth relations are a weak dual equivalence on reduced words for w.



Thank You