Rogers - Ramanujan identities

& Cylindric partitions

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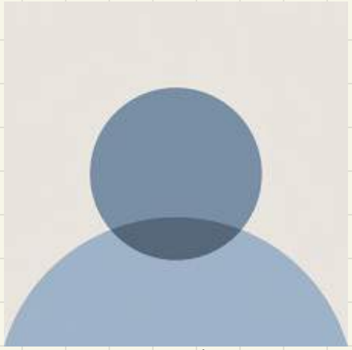
May 2022

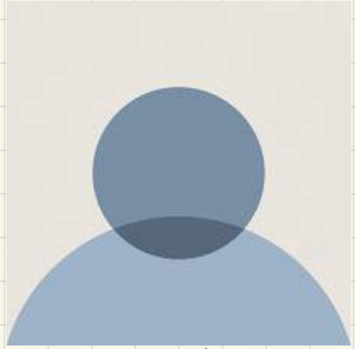


T. Welsh

O. Foda

J. Phys A. (2016)

S.C.

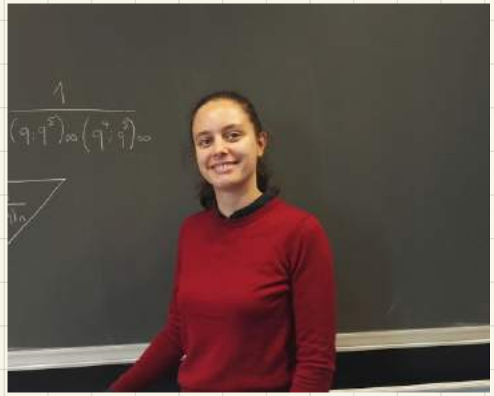


T. Welsh

Ann Comb (2019)

Proc of the AMS (2017)

S.C.



J. Dousse

A. Uncu

Proc of the AMS (2021)

O. Warnaar

Arxiv (2021)

S. Kanade

Arxiv (2022)

M. Russell

S. Tsuchioka

Arxiv (2022)

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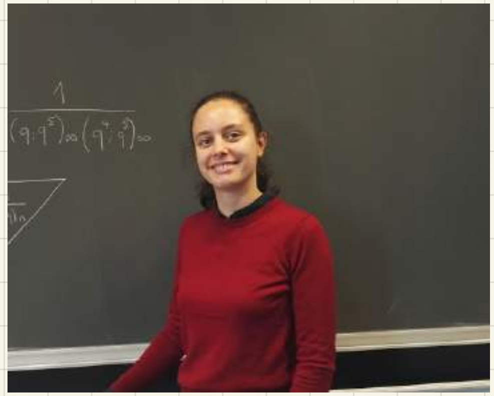

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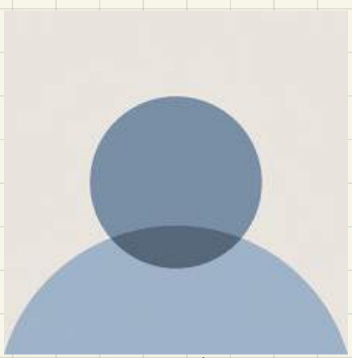
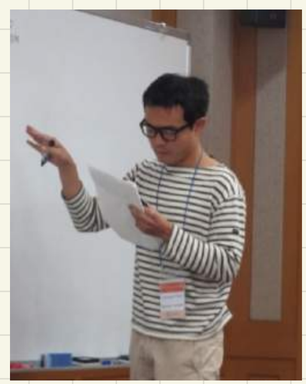


T. Welsh



S. Tsuchioka

M. Russell

Integer partitions

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$

Partition of N if $\lambda_1 + \lambda_2 + \lambda_3 + \ldots = N$

$(4,3,1)$ is a partition of 8.

Integer partitions

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$

Partition of N if $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \ldots = N$

$$\sum_{\lambda \,|\, \lambda_i \in S} q^{|\lambda|} = \prod_{i \in S} \frac{1}{1-q^i}$$

Integer partitions

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

such that $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots$

Interlacing partitions $\lambda \succcurlyeq \mu$

if

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \mu_2 \geqslant \lambda_3 \geqslant \mu_3 \geqslant \cdots$$

ex $(4,3,1) \succcurlyeq (4,1,1)$

$$4 \geqslant 4 \geqslant 3 \geqslant 1 \geqslant 1 \geqslant 1$$

Integer partitions

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$$

such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$

Interlacing partitions $\lambda \succcurlyeq \mu$

if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \dots$$

$$\sum_{\lambda^{(n)} \succcurlyeq \dots \succcurlyeq \lambda^{(1)} \succcurlyeq \emptyset} \prod_{i=1}^{n} x_i^{|\lambda_i| - |\lambda_{i-1}|} = s_{\lambda^{(n)}}(x_1, \dots, x_n)$$

Rogers – Ramanujan identities (1910s)

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

$$(a;q)_\infty = \prod_{i\geq 0} (1-aq^i) \qquad (a;q)_n = \prod_{i=0}^{n-1} (1-aq^i)$$

Rogers – Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}}_{\text{Sum}} = \underbrace{\frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}}_{\text{Product}}$$

Rogers – Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}}_{\text{Sum}} = \underbrace{\frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}}_{\text{Product}}$$

Sum = Product

Fermion Boson

Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}}_{\text{“Difference condition”}} = \underbrace{\frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}}_{\text{“Congruence classes”}}$$

“Difference condition” $\longleftrightarrow$ “Congruence classes”

Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}} = \underbrace{\frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}}$$

"Difference condition" $\overset{?}{\longleftrightarrow}$ "Congruence classes"

λ partition of N

$\lambda_i - \lambda_{i+1} \geq 2$

μ partition of N

$\mu_i \equiv 2$ or $3 \bmod 5$

Ramanujan identities

$$\underbrace{\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}} = \underbrace{\frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}}$$

"Difference condition" $\overset{?}{\longleftrightarrow}$ "Congruence classes"

λ partition of N

$\lambda_i - \lambda_{i+1} \geq 2$, no one

μ partition of N

$\mu_i \equiv 2$ or 3 mod 5

$N = 12$ (10) (8,2) (7,3) (6,4)

(8,2) (7,3) (3,3,2,2) (2^5)

Ramanujan identities (TODAY)

$$\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q;q)_\infty}\cdot\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

Ramanujan identities (TODAY)

$$\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n} = \underbrace{\frac{1}{(q;q)_\infty}\cdot\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}}_{\substack{\text{Cylindric partitions of}\\ \text{profile } (3,0)\\ \text{(Borodin 2007)}}}$$

Ramanujan identities (TODAY)

$$\underbrace{\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}}_{\text{Cylindric partitions of profile }(3,0)\text{ (Foda \& Welsh 2016)}} = \frac{1}{(q;q)_\infty}\cdot\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

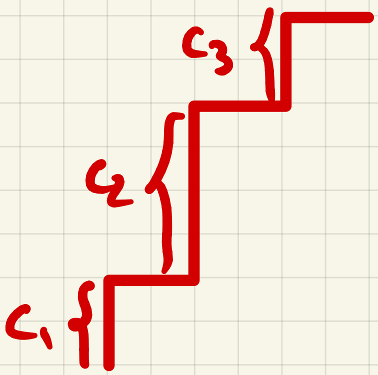
Ramanujan identities (TODAY)

$$\frac{1}{(q;q)_\infty}\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q;q)_\infty}\cdot\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

Cylindric partitions of profile $(3,0)$

Cylindric partitions (Gessel & Krattenthaler) $d, r > 0$

c

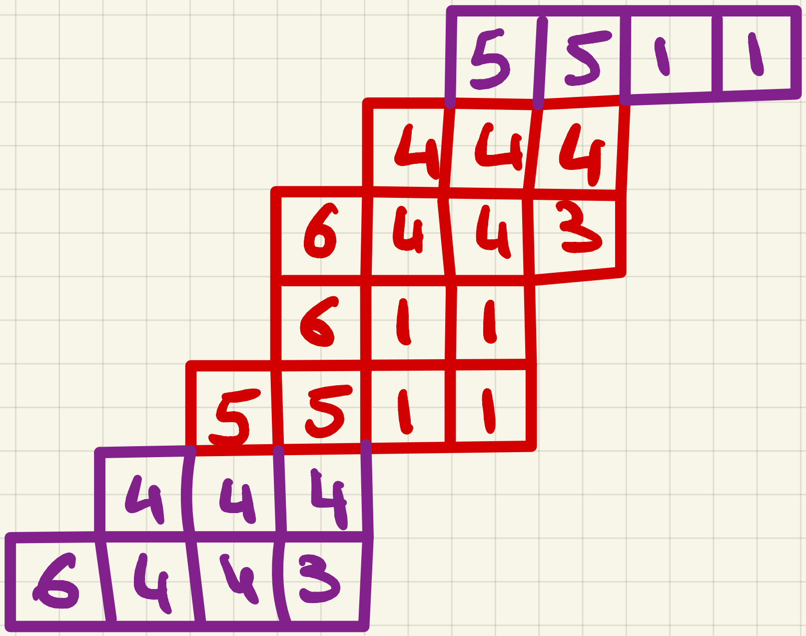


c a composition $(c_1, c_2, \ldots, c_r)$

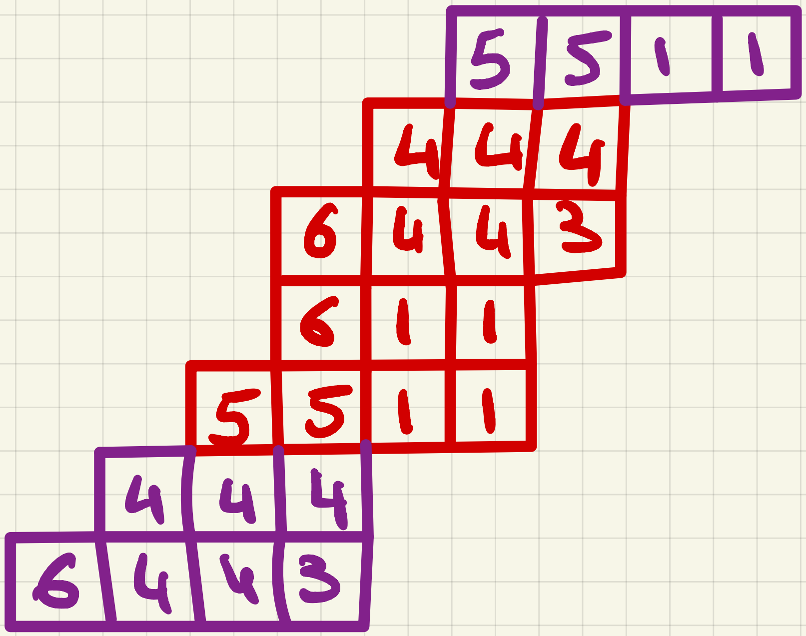
$c_1 + c_2 + \ldots + c_r = d$

ex $r = 3$ $d = 4$ $c = (1,2,1)$

Cylindric partitions



Cylindric partitions

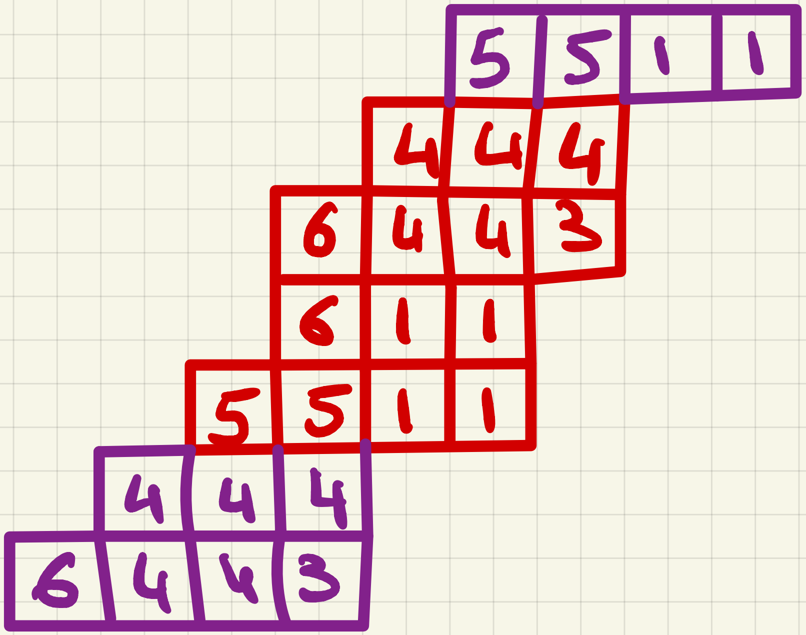


Cylindric partitions of profile $(1,2,1)$

$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \succcurlyeq \lambda^{(2)} \preccurlyeq \lambda^{(3)} \preccurlyeq \lambda^{(4)} \succcurlyeq \lambda^{(5)} \preccurlyeq \lambda^{(6)} \succcurlyeq \lambda^{(7)}$$

$$\lambda^{(0)} = \lambda^{(7)}$$

Cylindric partitions



c a composition $(c_1, c_2, \ldots, c_r)$

$\downarrow$ interlacing word

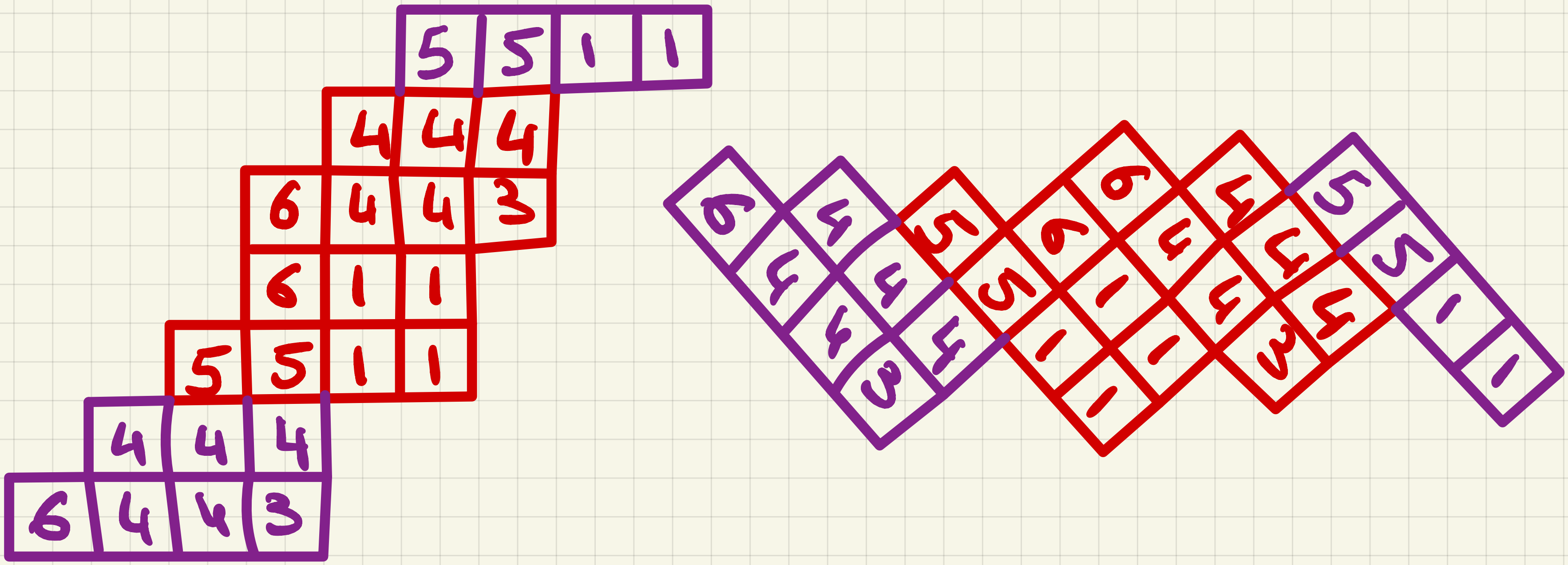
$$(\preccurlyeq)^{c_1} \succcurlyeq (\preccurlyeq)^{c_2} \succcurlyeq \cdots$$

Cylindric partitions of profile $(1,2,1)$

$$\lambda^{(0)} \preccurlyeq \lambda^{(1)} \succcurlyeq \lambda^{(2)} \preccurlyeq \lambda^{(3)} \preccurlyeq \lambda^{(4)} \succcurlyeq \lambda^{(5)} \preccurlyeq \lambda^{(6)} \succcurlyeq \lambda^{(7)}$$

$$\lambda^{(0)} = \lambda^{(7)}$$

Cylindric partitions



Cylindric partitions of profile $(1,2,1)$

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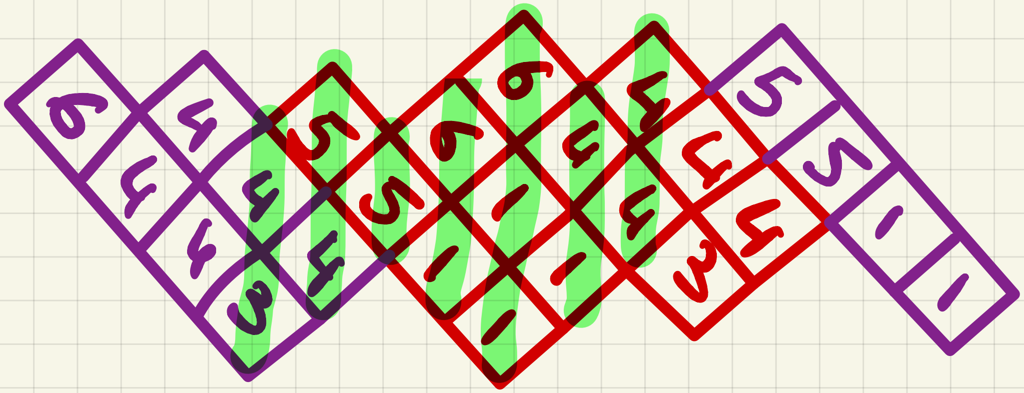
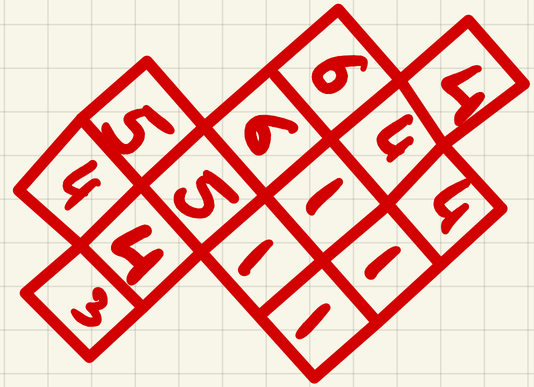
Cylindric partitions





Cylindric partitions of profile $(1,2,1)$

$$\lambda^{(0)} \preceq \lambda^{(1)} \succeq \lambda^{(2)} \preceq \lambda^{(3)} \preceq \lambda^{(4)} \succeq \lambda^{(5)} \preceq \lambda^{(6)} \succeq \lambda^{(7)} = \lambda^{(0)}$$

$$(4,3) \preceq (5,4) \succeq (5) \preceq (6,1) \preceq (6,1,1)$$
$$\succeq (4,1) \preceq (4,4) \succeq (4,3)$$

Theorem (Borodin 2007) $d+r=t$

$$\sum_{\substack{\Lambda \text{ a cylindric} \\ \text{of profile } c}} q^{|\Lambda|} = \text{Beautiful product involving hooks}$$

$$= \frac{1}{(q^t;q^t)_\infty} \prod_{n\geq 0} \prod_{\square \in c} \frac{1}{1-q^{nt+\mathrm{cylhook}(\square)}}$$

Theorem (Borodin)

$$\sum_{\substack{\text{a cylindric}\\ \text{of profile } c}} q^{|\Lambda|} = \text{Beautiful product involving hooks}$$

ex $c = (2,1)$

$$\text{Product} = \frac{1}{(q;q)_\infty (q,q^4;q^5)_\infty}$$

$$(a_1,\dots,a_k;q)_\infty = \prod_{i=1}^{k}\prod_{j=1}^{\infty}(1-a_i q^j)$$

Theorem (Borodin)

$$\sum_{\substack{\lambda \text{ cylindric} \\ \text{of profile } c}} q^{|\lambda|} = \text{Beautiful product involving hooks}$$

ex $c = (2,1)$

$$\text{Product} = \frac{1}{(q;q)_\infty \, (q, q^4; q^5)_\infty}$$

$$(a_1, \dots, a_k; q)_n = \prod_{i=1}^{k} \prod_{j=0}^{n-1} (1 - a_i q^j)$$

Rogers – Ramanujan identity

$$\sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q, q^4 ; q^5)_\infty}$$

Theorem (Borodin)

$$\sum_{\substack{\text{a cylindric} \\ \text{of profile} \\ w_1 \dots w_T}} q^{|\lambda|} = \text{Beautiful product involving hooks}$$

ex $\quad c = (3,0)$

$$\text{Product} = \frac{1}{(q;q)_\infty (q^2, q^3; q^5)_\infty}$$

$$(a_1, \dots, a_k; q)_n = \prod_{i=1}^{k} \prod_{j=0}^{n-1} (1 - a_i q^j)$$

Rogers - Ramanujan identity II

$$\sum_{n \geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2, q^3 ; q^5)_\infty}$$

<u>Foda & Welsh</u> (2016) $r \geq 1$ profile $c = (c_1, \dots, c_r)$

Generating function of cylindric partitions $\longleftrightarrow$

Character formula W_r-algebra

Foda & Welsh $r \geq 1$ profile $c = (c_1, \ldots, c_r)$

Character formula W_r-algebra

$r = 2$ $c = (i, d - i)$

product $$\frac{(q^{d+2}, q^{i+1}, q^{d-i+1}; q^{d+2})_\infty}{(q;q)_\infty^2}$$

$i = 1, \ldots, k$

related to Andrews - Gordon identities
Bressoud

Andrews - Gordon - Bressoud identities

$d = 2k+1$

$$\sum_{n_1, \ldots, n_{k-1}} \frac{q^{n_1^2 + \ldots + n_{k-1}^2 + n_i + \ldots + n_{k-1}}}{(q)_{n_1 - n_2} (q)_{n_2 - n_3} \cdots (q)_{n_{k-2} - n_{k-1}}} \frac{1}{(q)_{n_{k-1}}}$$

$$= \frac{(q^{2k+1}, q^i, q^{2k+1-i}; q^{2k+1})_\infty}{(q)_\infty}$$

Andrews - Gordon - Bressoud identities

$d = 2k+1$

$$\sum_{n_1,\dots,n_{k-1}} \frac{q^{n_1^2+\dots+n_{k-1}^2+n_i+\dots+n_{k-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{k-2}-n_{k-1}}} \frac{1}{(q)_{n_{k-1}}} = \frac{(q^{2k+1}, q^{i}, q^{2k+1-i}; q^{2k+1})_\infty}{(q)_\infty}$$

$d = 2k$

$$\sum_{n_1,\dots,n_{k-1}} \frac{q^{n_1^2+\dots+n_{k-1}^2+n_i+\dots+n_{k-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3}\cdots(q)_{n_{k-2}-n_{k-1}}} \frac{1}{(q^2;q^2)_{n_k}} = \frac{(q^{2k}, q^{i}, q^{2k-i}; q^{2k})_\infty}{(q)_\infty}$$

Again the cylindric partitions give

$$\frac{1}{(q;q)_\infty} \cdot \text{Product of the A-G-B identities.}$$

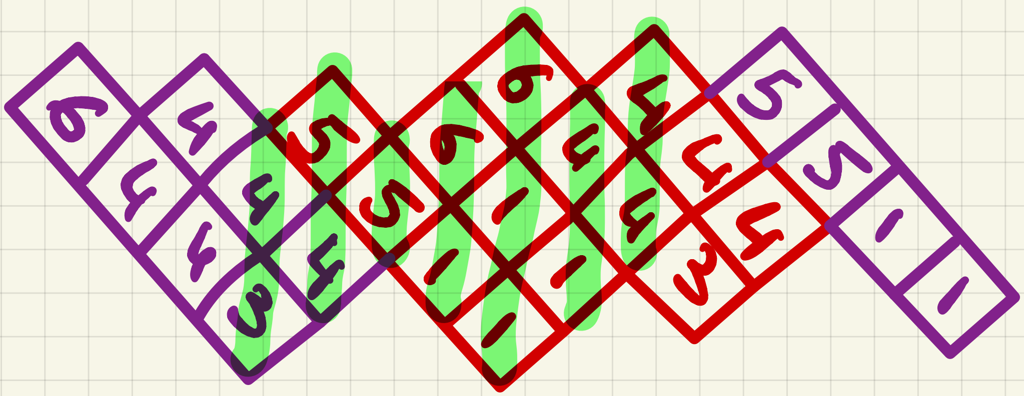
<u>Q</u> : How about $r \geqslant 3$?

Conjecture (Foda & Welsh)

There exists a Rogers–Ramanujan identity for each composition $(c_1, \ldots, c_r)$ of d

for all r & d

"Up to rotation and conjugation"



Rotation

$$c = (1,2,1)$$

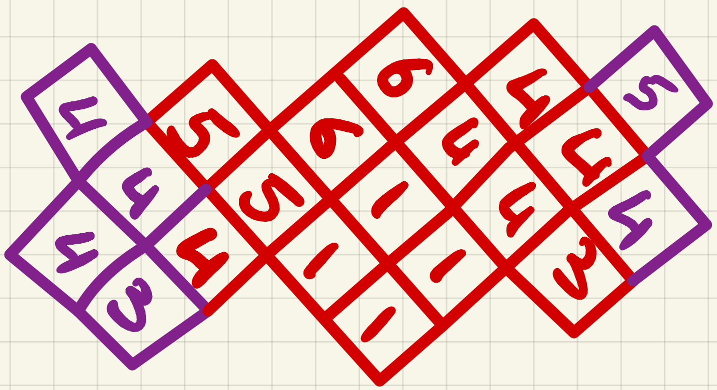
$$\updownarrow$$

$$\tilde{c} = (2,1,1)$$

$$\updownarrow$$

$$\tilde{\tilde{c}} = (1,1,2)$$

"Up to rotation and conjugation"



Rotation

$c = (1,2,1)$

$\updownarrow$

$c^r = (2,1,1)$

$\updownarrow$

$c^{rr} = (1,1,2)$

Conjugation

$c = (1,2,1) \longleftrightarrow c' = (1,1,0,1)$

Main Tool (Corteel & Welsh 2019)

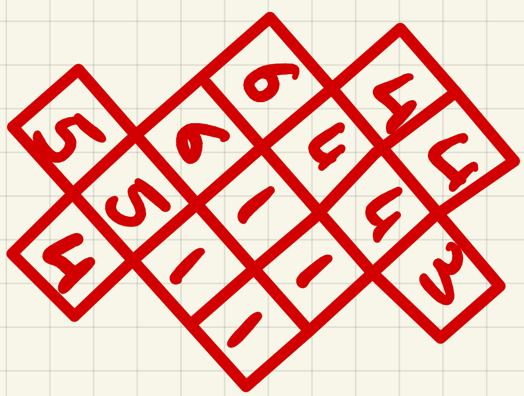
$$c = (c_1, \dots, c_r)$$

$$F_c(z;q) = \sum_{\lambda \text{ profile } c} z^{\max(\lambda)} q^{|\lambda|}$$

<u>Main Tool</u> (Corteel & Welsh)

$$c = (c_1, \dots, c_r)$$

$$F_c(z;q) = \sum_{\Lambda \text{ profile } c} z^{\max(\Lambda)} \, q^{|\Lambda|}$$



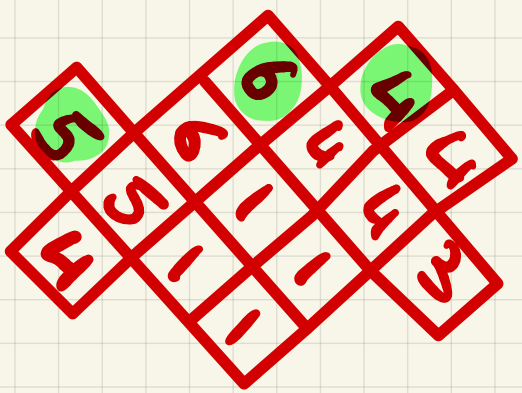
$$|\Lambda| = 49$$

$$\max(\Lambda) = 6$$

Main Tool (Corteel & Welsh)

$$c = (c_1, \dots, c_n)$$

$$F_c(z;q) = \sum_{\lambda \text{ profile } c} z^{\max(\lambda)} q^{|\lambda|}$$



Theorem

$$F_c(z,q) = \sum_{S \text{ subset of the corners}} (-1)^{|S|} \frac{F_{c(S)}(zq^{|S|},q)}{1-zq^{|S|}}$$

Example

$$F_{30}(z,q) = \frac{F_{2,1}(zq,q)}{(1-zq)}$$

$$F_{21}(z,q) = \frac{F_{30}(zq,q)}{(1-zq)} + \frac{F_{21}(zq,q)}{(1-zq)} - \frac{F_{21}(zq^2,q)}{(1-zq^2)}$$

Example

$$F_{30}(z,q) = \frac{F_{2,1}(zq,q)}{(1-zq)}$$

$$F_{21}(z,q) = \frac{F_{30}(zq,q)}{(1-zq)} + \frac{F_{21}(zq,q)}{(1-zq)} - \frac{F_{21}(zq^2,q)}{(1-zq^2)}$$

Lemma

$$F_{30}(z,q) = \frac{1}{(zq;q)_\infty} \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q;q)_n}$$

$$F_{21}(z,q) = \frac{1}{(zq;q)_\infty} \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n}$$

Example

$$F_{30}(z,q) = \frac{F_{2,1}(zq,q)}{(1-zq)}$$

$$F_{21}(z,q) = \frac{F_{30}(zq,q)}{(1-zq)} + \frac{F_{21}(zq,q)}{(1-zq)} - \frac{F_{21}(zq^2,q)}{(1-zq^2)}$$

Lemma

$$F_{30}(z,q) = \frac{1}{(zq;q)_\infty} \sum_{n\geq 0} \frac{z^n q^{n^2+n}}{(q;q)_n}$$

$$F_{21}(z,q) = \frac{1}{(zq;q)_\infty} \sum_{n\geq 0} \frac{z^n q^{n^2}}{(q;q)_n}$$

Conjecture (Corteel & Welsh, Warnaar)

$c = (c_1, \ldots, c_r)$ $\quad c_1 + \ldots + c_r = d$ $\quad k = \mathrm{gdc}(d, r)$

$$F_c(z, q) = \frac{1}{(zq;q)_\infty} \sum_n \frac{z^n}{(q^k;q^k)_n} F_{c,n}(q)$$

Conjecture (Corteel & Welsh, Warnaar)

$c = (c_1, \dots, c_n)$ $\quad c_1 + \dots + c_n = r$ $\quad d = \gcd(n, r)$

$$F_c(z,q) = \frac{1}{(zq;q)_\infty} \sum_{n_1} \frac{z^n}{(q^k;q^k)_n} F_{c,n}(q)$$

$F_{c,n}(q)$ is a polynomial into non negative coefficients

$$P_{c,n}(1) = k^n \left(\frac{1}{d+r} \binom{d+r}{r} - 1 \right)^n$$

What is known so far?

- $r = 1$ ✓
- $r = 2$ ✓
- $r = 3$ $d = 2, 4, 5$

(C. 2016, C & Welsh 2019
C. Dousse & Uncu 2021
Warnaar 2021)

$d = 3$ (Tsuchioka 2022)

What is known so far?

- $n = 1$
- $n = 2$
- $n = 3$ $\quad r = 2, 4, 5$

(C. 2016, C & Welsh 2019
C. Dousse & Uncu 2021
Warnaar 2021)

$r = 3$ (Tsuchioka 2022)

Conjecture (Warnaar 2021) $\quad k > 0$

$c = (k, k-1, k-1)$ $\qquad c = (3k-s, s-1, 0)$

$c = (k, k, k-1)$ $\qquad c = (3k-s-1, s-1, 0)$

$1 \leq s \leq k+1$

Conjecture (Warnaar)

$$F_{(k,k-1,k-1),n,}(q) = \sum_{\substack{n_2,\ldots,n_k \\ m_1,\ldots,m_{k-1}}} q^{n_k^2} \prod_{i=1}^{k-1} q^{n_i^2+m_i^2-n_i m_i} \begin{bmatrix} n_i \\ m_{i+1} \end{bmatrix}_q \begin{bmatrix} n_i-n_{i+1}+m_{i+1} \\ m_i \end{bmatrix}_q$$

Conjecture (Warnaar 21)

$$F_{(k,k-1,k-1),n_1}(q) = \sum_{\substack{n_2,\dots,n_k \\ m_1,\dots,m_{k-1}}} q^{n_k^2} \prod_{i=1}^{k-1} q^{n_i^2+m_i^2-n_i m_i} \begin{bmatrix} n_i \\ m_{i+1} \end{bmatrix}_q \begin{bmatrix} n_i-n_{i+1}+m_{i+1} \\ m_i \end{bmatrix}_q$$

$$P_{(k,k\ \ ,k-1),n_1}(1) = \left(\frac{1}{3k+1} \binom{3k+1}{3} - 1 \right)^{n_1}$$

C., Welsh $\quad F_c(z,q) = \dfrac{1}{(zq;q)_\infty} G_c(z,q)$

New approach $\quad$ Kanade & Russell (2022)

$$F_c(z,q) = \frac{(zq,q)_\infty}{(q;q)_\infty} H_c(z,q)$$

Our approach $F_c(z,q) = \frac{1}{(zq;q)_\infty} G_c(z,q)$

New approach Kanade & Russell (2022)

$$F_c(z,q) = \frac{(zq,q)_\infty}{(q;q)_\infty} H_c(z,q)$$

$r=3$ $\quad c=(k-s,\ s,\ 0)$ $\quad 0 \leq s < \frac{k}{3}$

Andrews, Schilling, Warnaar (2002)

$r=3$ all compositions of $d \leq 7$ (Kanade & Russell 2021)

all compositions of 8 (Uncu 2022)

Proof techniques

- Combinatorics
- q - series
- Computer algebra

Proof techniques

$$
\begin{aligned}
G_{(5,0,0)}(z,q) &= G_{(4,1,0)}(zq,q),\\
G_{(4,1,0)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(3,2,0)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(4,0,1)}(z,q) &= G_{(5,0,0)}(zq,q) + G_{(3,1,1)}(zq,q) - (1-zq)G_{(4,1,0)}(zq^2,q),\\
G_{(3,2,0)}(z,q) &= G_{(3,1,1)}(zq,q) + G_{(3,0,2)}(zq,q) - (1-zq)G_{(2,2,1)}(zq^2,q),\\
G_{(3,1,1)}(z,q) &= G_{(4,1,0)}(zq,q) + G_{(3,0,2)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(4,0,1)}(zq^2,q) + G_{(3,2,0)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(3,1,1)}(zq^3,q),\\
G_{(3,0,2)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(2,2,1)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(2,2,1)}(z,q) &= G_{(3,2,0)}(zq,q) + G_{(3,1,1)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(3,1,1)}(zq^2,q) + G_{(3,0,2)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(2,2,1)}(zq^3,q).
\end{aligned}
$$

C. Dousse & Uncu   Automatic proofs

$$
\begin{aligned}
G_{(5,0,0)}(z,q) &= G_{(4,1,0)}(zq,q),\\
G_{(4,1,0)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(3,2,0)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(4,0,1)}(z,q) &= G_{(5,0,0)}(zq,q) + G_{(3,1,1)}(zq,q) - (1-zq)G_{(4,1,0)}(zq^2,q),\\
G_{(3,2,0)}(z,q) &= G_{(3,1,1)}(zq,q) + G_{(3,0,2)}(zq,q) - (1-zq)G_{(2,2,1)}(zq^2,q),\\
G_{(3,1,1)}(z,q) &= G_{(4,1,0)}(zq,q) + G_{(3,0,2)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(4,0,1)}(zq^2,q) + G_{(3,2,0)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(3,1,1)}(zq^3,q),\\
G_{(3,0,2)}(z,q) &= G_{(4,0,1)}(zq,q) + G_{(2,2,1)}(zq,q) - (1-zq)G_{(3,1,1)}(zq^2,q),\\
G_{(2,2,1)}(z,q) &= G_{(3,2,0)}(zq,q) + G_{(3,1,1)}(zq,q) + G_{(2,2,1)}(zq,q)\\
&\quad - (1-zq)\big(G_{(3,1,1)}(zq^2,q) + G_{(3,0,2)}(zq^2,q) + G_{(2,2,1)}(zq^2,q)\big)\\
&\quad + (1-zq)(1-zq^2)G_{(2,2,1)}(zq^3,q).
\end{aligned}
$$

Theorem (CDU, 2021)

$$
\sum_{n_1,n_2,n_3,n_4\geq 0} \frac{q^{n_1^2+n_2^2+n_3^2+n_4^2-n_1n_2+n_2n_4}}{(q;q)_{n_1}} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}_q \begin{bmatrix} n_1 \\ n_4 \end{bmatrix}_q \begin{bmatrix} n_2 \\ n_3 \end{bmatrix}_q = \frac{1}{(q,q,q^2,q^4,q^4,q^6,q^7,q^7;q^8)_\infty}.
$$

Automatic proof

Proof techniques

- Combinatorics
- q - series
- Computer algebra
- Lie theory

Q : Can we combine all those to prove identities for $c = (c_1, \ldots, c_r)$ compositions of d ?

A lovely special case $\quad d = r+1$

Conjecture $(c_1, \ldots, c_r)$

There are C_r RR identities and they are of the form

$$\frac{1}{(q)_\infty} \sum_n \frac{F_{c,n}(q)}{(q;q)_{n_1}} = \prod_{\square \in c} \frac{1}{1 - q^{\mathrm{cychook}(\square)}}$$

A lovely special case $d = r+1$

Conjecture $(c_1, \ldots, c_r)$

Catalan

There are C_r RR identities and they are of the form

$$\frac{1}{(q)_\infty} \sum_n \frac{F_{c,n}(q)}{(q;q)_n} = \prod_{\square \in c} \frac{1}{1 - q^{\mathrm{cychook}(\square)}}$$

$F_{c,n}(q) \in \mathbb{N}[q]$ $\qquad F_{c,n}(q) = (C_r - 1)^n$

A lovely special case $d = r+1$

Conjecture $(c_1, \ldots, c_r)$ Catalan

There are C_r RR identities and they are of the form

$$\frac{1}{(q)_\infty} \sum_n \frac{F_{c,n}(q)}{(q;q)_n} = \prod_{\square \in c} \frac{1}{1 - q^{\mathrm{cychook}(\square)}}$$

$$F_{c,n}(q) = (C_r - 1)^n$$

Open for $r \geq 4$

$r = 1$ $\qquad C_1 = 1$ $\qquad F_{(2),n}(q) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

$r = 2$ $\qquad C_2 = 2$

$$F_{(3,0),n}(q) = q^{n^2+n}$$

$$F_{(2,1),n}(q) = q^{n^2}$$

$r = 3$ $\qquad C_3 = 5$

$$F_{(4,0,0),n}(q) = \sum_{n_2} q^{n^2 - n n_2 + n_2^2 + n + n_2} \begin{bmatrix} 2n \\ n_2 \end{bmatrix}$$

$$F_{(3,1),n}(q) = \sum_{n_2} q^{n^2 - n n_2 + n_2^2 + n + n_2} \begin{bmatrix} 2n \\ n_2 \end{bmatrix}$$

$$\vdots$$

$$F_{(2,2,1),n}(q) = \sum_{n_2} q^{n^2 - n n_2 + n_2^2 + n + n_2} \begin{bmatrix} 2n \\ n_2 \end{bmatrix}$$

$r = 4$ $\qquad ?$

Q : How to guess those multisums ?

Combinatorics : could we find more statistics on cylindric partitions

1^{st} sum $\longleftrightarrow$ $\max(\Lambda)$

2^{nd} sum $\longleftrightarrow$??

$\vdots$