Generalized q, t-Catalan polynomials and link invariants

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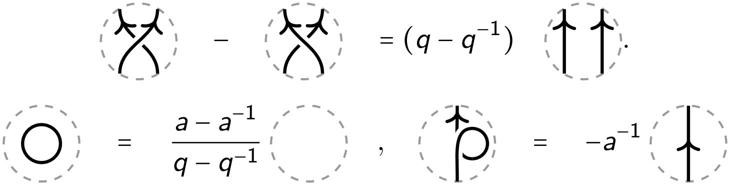
Khovanov and Rozansky defined in 2005 a triply graded link homology theory which generalizes HOMFLY-PT polynomial. In this talk, I will describe the progress in understanding this homology, focusing on:

- Examples of computations of Khovanov-Rozansky homology
- Connections to q, t-Catalan combinatorics
- General structures in Khovanov-Rozansky homology
- Geometric models for some classes of links.

HOMFLY-PT invariant

The HOMFLY-PT invariant of links is defined by the following rules:



Given an (oriented) link diagram in the plane, we can use these rules to simplify it until the link becomes trivial.

HOMFLY homology

Khovanov and Rozansky defined HOMFLY homology and proved that it is a link invariant. To any link they assign a triply graded vector space $\mathcal{H} = \bigoplus_{i,j,k} \mathcal{H}_{i,j,k}$ such that the graded Euler characteristic

$$\sum_{i,j,k} q^i a^j (-1)^k \dim \mathcal{H}_{i,j,k} = P(a,q)$$

recovers the HOMFLY-PT polynomial.

The definition of Khovanov-Rozansky homology is quite involved and uses **Hochschild homology** for complexes of **Soergel bimodules**. We will not need it.

HOMFLY-PT homology: examples

Here is the (3, 4) torus knot[1] and its Khovanov-Rozansky homology[2]:

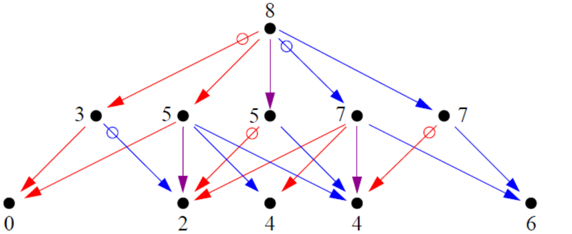


FIGURE 3.7. Differentials for $T_{3,4}$.The bottom row of dots has a-grading 6. The leftmost dot on that row has q-grading -6, which you can determine by noting that the vertical axis of symmetry corresponds to the line $q = 0$.

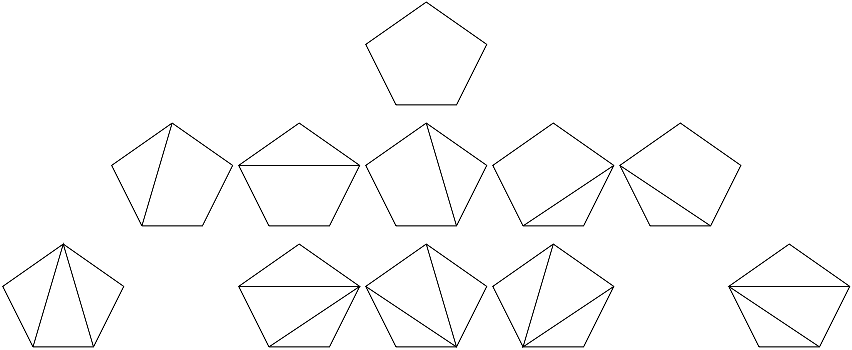
Each dot represents a generator in Khovanov-Rozansky homology, so the total dimension of homology is $5 + 5 + 1 = 11$.

[1] Picture credit: The Knot Atlas

[2] Picture credit: S. Gukov, N. Dunfield, J. Rasmussen

HOMFLY-PT homology: examples

Observe that there are **5** ways to draw two non-intersecting diagonals in a pentagon, **5** ways to draw one diagonal. and **1** way to draw no diagonals, in total $5 + 5 + 1 = 11$.



The number of triangulations of an *n*-gon is called the **Catalan number**.

HOMFLY-PT homology: examples

The following result was proved by Hogancamp and Mellit in 2017, following my conjecture from 2010:

Theorem (Hogancamp, Mellit)

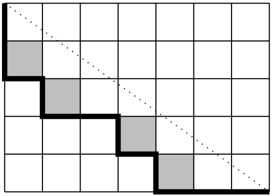
a) The total dimension of HOMFLY homology for the $(n, n+1)$ torus knot equals the number of ways to draw non-intersecting diagonals in the $(n+2)$-gon.
b) The bigraded dimension of the "bottom row" of HOMFLY homology for the $(n, n+1)$ torus knot equals the q, t-Catalan number $c_n(q, t)$.
c) More generally, the bigraded dimension of the "bottom row" of HOMFLY homology for the (m, n) torus knot equals the corresponding rational q, t-Catalan number $c_{m,n}(q, t)$.

q, t-Catalan numbers

The q, t-Catalan numbers and their rational analogues were introduced and studied by Bergeron, Garsia, Haiman, Haglund and many others. They can be defined as

$$c_{m,n}(q,t) = \sum_{D} q^{\mathrm{area}(D)} t^{\mathrm{dinv}(D)}$$

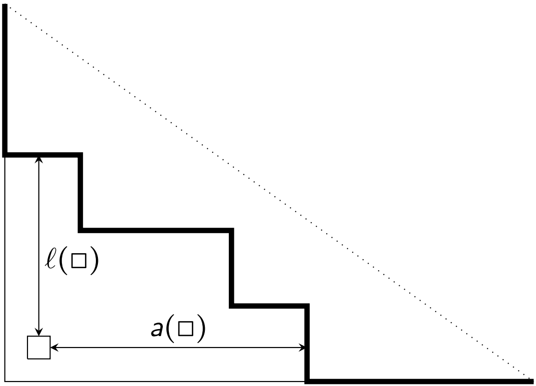
where the sum is over all lattice paths D in the $m \times n$ rectangle which do not cross the diagonal, and $\mathrm{area}(D), \mathrm{dinv}(D)$ are certain combinatorial statistics. Here $\mathrm{area}(D) = 4$:



q, t-Catalan numbers

The definition of $\operatorname{dinv}(D)$ is more complicated:

$$\operatorname{dinv}(D) = \sharp \left\{ \square \in D : \frac{a(\square)}{\ell(\square)+1} < \frac{m}{n} < \frac{a(\square)+1}{\ell(\square)} \right\}$$



q,t-Catalan numbers

The following theorem is a special case of "Rational Shuffle Conjecture" of G.-Neguț,Bergeron-Garsia-Leven-Xin:

Theorem (Mellit)

Suppose that $GCD(m,n)=1$. We have

$$c_{m,n}(q,t)=\sum_D q^{\mathrm{area}(D)}t^{\mathrm{dinv}(D)}=$$

$$\sum_{T\in\mathrm{SYT}(n)}\frac{z_1^{d_1}\cdots z_n^{d_n}}{(1-z_i^{-1})(1-qtz_i/z_{i+1})}\prod_{i<j}\frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}$$

where the sum is over all standard Young tableaux T of size n, z_i is the (q,t)-content of a box labeled by i in T and $d_i=\lceil\frac{im}{n}\rceil-\lceil\frac{(i-1)m}{n}\rceil$.

As a consequence, the left hand side is symmetric in q and t. The right hand side is a polynomial in q and t with nonnegative coefficients, and symmetric in m and n.

q, t-Catalan numbers

The q, t-Catalan numbers are related to:

- Combinatorics of Macdonald polynomials and Shuffle Conjecture
- Geometry of the Hilbert scheme of points on the plane
- Representation theory of DAHA and Elliptic Hall Algebra

Overall, there are several very different ways to compute $c_{m,n}(q,t)$, and hence the HOMFLY homology of torus knots are known.

Problem

The **generalized** q, t-**Catalan numbers** *are given by the above formula with arbitrary* $d_1, \ldots, d_n$. *How to understand them combinatorically? Conjecturally, for* $d_1 \geq d_2 \cdots \geq d_n$ *these have nonnegative coefficients and correspond to HOMFLY homology of certain knots.*

There are few other classes of knots with known HOMFLY homology. Nakagane and Sano recently computed HOMFLY homology for all knots with at most 10 crossings, and most 11-crossing knots, so there is a lot of data to be explored.

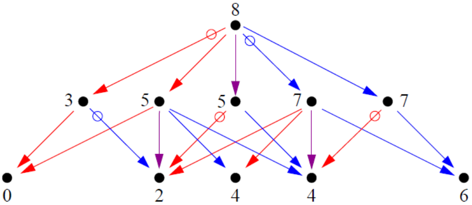
HOMFLY-PT homology: structures

The following result was conjectured by Gukov, Dunfield and Rasmussen in 2005, but took very long time to prove.

Theorem (G., Hogancamp, Mellit, 2021)

The HOMFLY homology of any knot is symmetric around the vertical axis. Furthermore, there is an action of the Lie algebra $\mathfrak{sl}(2)$ in HOMFLY homology which yields this symmetry.

Related results were obtained by Galashin-Lam and Oblomkov-Rozansky.



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HOMFLY-PT homology: structures

For torus knots, the symmetry implies $c_{m,n}(q,t) = c_{m,n}(t,q)$, which is a highly nontrivial combinatorial identity.

For links with several (say, r) components, the HOMFLY homology is infinite-dimensional and it is a module over the polynomial ring $\mathbb{C}[x_1, \ldots, x_r]$. Hogancamp and I defined a deformation, or y-ification of HOMFLY homology for links which depends on additional variables $y_1, \ldots, y_r$.

Theorem (G., Hogancamp, Mellit, 2021)

The y-ified homology of any link is symmetric, and the symmetry exchanges x_i with y_i.

HOMFLY-PT homology: structures

For example, consider the (n, n) torus link with n unknotted components which are pairwise linked.

Theorem (G., Hogancamp, 2017)

a) The "bottom row" of the y-ified homology of the (n, n) torus link is isomorphic to the ideal

$$J = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j) \subset \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

which is the defining ideal for the union of diagonals in $(\mathbb{C}^2)^n$.
b) The HOMFLY homology of the (n, n) torus link is isomorphic to $J/(y_1, \ldots, y_n)J$.

HOMFLY-PT homology: geometric models

The **first** geometric model for HOMFLY homology is given by **braid varieties**. Recall that the braid group on the n strands is defined by generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i-j| > 1).$$

We define matrices

$$B_i(z) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & z & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

One can check that
$B_i(z_1) B_{i+1}(z_2) B_i(z_3) = B_{i+1}(z_3) B_i(z_2 - z_1 z_3) B_{i+1}(z_1)$.

HOMFLY-PT homology: geometric models

Given a braid $\beta = \sigma_{i_1} \cdots \sigma_{i_k}$, we define the braid variety

$$X(\beta) = \left\{ z_1, \ldots, z_k \; : \; B_{i_1}(z_1) \cdots B_{i_k}(z_k) \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \quad \text{upper} - \text{triangular} \right\}$$

Theorem (Escobar; Casals, G., M. Gorsky, Simental)

$X(\beta)$ is either empty or it is a smooth manifold of dimension $k - \binom{n}{2}$. If β closes to a knot then $X(\beta) = (\mathbb{C}^*)^{n-1} \times Y(\beta)$ for some $Y(\beta)$.

Theorem (Webster-Williamson, Mellit, Trinh)

Suppose that β closes to a knot. The "bottom row" of HOMFLY homology is isomorphic to the homology of $Y(\beta)$ equipped with **weight filtration**.

HOMFLY-PT homology: geometric models

Example

For $\beta = \sigma_1^3$ we have $Y(\beta) = \{z_1, z_2, z_3 : z_1 + z_3 + z_1 z_2 z_3 = 1\} \subset \mathbb{C}^3$.

Theorem (Galashin, Lam, 2021)

For torus knots, the variety $X(\beta)$ is isomorphic (up to $(\mathbb{C}^*)^{\cdots}$) to the **open positroid variety** $\Pi^\circ_{m,n} \subset Gr(m, n+m)$ *defined by the non-vanishing of cyclically consecuitive minors. The homology of $\Pi^\circ_{m,n}$ equipped with weight filtration are given by q, t-Catalan numbers $c_{m,n}(q,t)$.*

HOMFLY-PT homology: geometric models

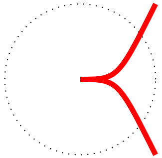
The **second** geometric model is given by **affine Springer fibers**. Let $\gamma(t)$ be an $n \times n$ matrix depending on a parameter t, define the affine Springer fiber

$$\mathrm{Sp}_\gamma := \{V \subset \mathbb{C}^n((t)) : tV \subset V, \gamma(t)V \subset V\}.$$

This is an object of very active research in geometric representation theory.

Given such $\gamma(t)$, we can define the plane curve singularity $C = \{\det(\gamma(t) - y\mathrm{Id}) = 0\}$ and the knot $K = C \cap S^3_\epsilon$.

For example, $\gamma = \begin{pmatrix} 0 & t^3 \\ 1 & 0 \end{pmatrix}$ corresponds to $C = \{y^2 - t^3\}$ and to the $(2, 3)$ torus knot.



HOMFLY-PT homology: geometric models

For $\gamma(t) = \begin{pmatrix} 0 & & & t^m \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}$ the curve $C = \{y^n - t^m\}$ corresponds to the (m, n) torus knot.

Theorem (G., Mazin)

For such $\gamma(t)$ the homology of the affine Springer fiber agrees with the q, t-Catalan number.

Conjecture (Oblomkov, Rasmussen, Shende, 2012)

Under some mild assumptions on $\gamma(t)$, the homology of the affine Springer fiber Sp_γ is isomorphic to the HOMFLY homology of the link of C.

HOMFLY-PT homology: geometric models

The **third** geometric model is given by the **Hilbert scheme of points on the plane** $\mathrm{Hilb}^n(\mathbb{C}^2)$ which is the resolution of singularities of $(\mathbb{C}^2)^n/S_n$. Given a braid β, one expects a vector bundle (or a sheaf) $\mathcal{F}_\beta$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$ such that its space of sections (or sheaf cohomology) matches the HOMFLY homology of the link. The gradings correspond to the action of $(\mathbb{C}^*)^2$ on $\mathrm{Hilb}^n(\mathbb{C}^2)$.

Different (and, conjecturally, equivalent) constructions of $\mathcal{F}_\beta$ were proposed by G.-Neguț-Rasmussen, G.-Hogancamp and Oblomkov-Rozansky.

HOMFLY-PT homology: geometric models

Here are two motivating examples:

Example

The $(n, n+1)$ torus knot corresponds to the line bundle $\mathcal{O}(1)$ on the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$. By the work of Haiman, the bigraded dimension of its space of sections is given by the q,t-Catalan number.

Example

The (n, n) torus link corresponds to the vector bundle $\mathcal{P} \otimes \mathcal{O}(1)$ where $\mathcal{P}$ is the **Procesi bundle** of rank $n!$. By the work of Haiman, its space of sections is isomorphic to $J = \bigcap_{i \neq j}(x_i - x_j, y_i - y_j)$.

More examples, details and references:

E. Gorsky, M. Hogancamp, A. Mellit. Tautological classes and symmetry in Khovanov-Rozansky homology. arXiv:2103.01212

E. Gorsky, O. Kivinen, J. Simental. Algebra and geometry of link homology. arXiv:2108.10356

K. Nakagane, T. Sano. Computations of HOMFLY homology. arXiv:2111.00388

E. Gorsky, G. Hawkes, A. Schilling, J. Rainbolt. Generalized q, t-Catalan numbers. Algebraic Combinatorics 3(4) (2020) 855-886.

Thank You!