Structure constants: complexity and asymptotics

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Plethysm: Compositions of GL-representations.

$$S^d(S^n V) = \oplus_{\lambda \vdash dn} V_\lambda^{a_\lambda(d[n])}$$

Via Schur functions and S_n characters

$$s_\lambda(x,y) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu(x) s_\nu(y) \quad \Longleftrightarrow \quad s_{\lambda/\mu}(x) = \sum_{\nu} c^\lambda_{\mu,\nu} s_\nu(x)$$

$$s_\lambda[x.y] = \sum_{\mu,\nu} g(\lambda,\mu,\nu) s_\mu(x) s_\nu(y) \quad \Longleftrightarrow \quad \sum_{\lambda,\mu,\nu} g(\lambda,\mu,\nu) s_\lambda(x) s_\mu(y) s_\nu(z) = \prod_{i,j,k} \frac{1}{1 - x_i y_j z_k}$$

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$$\begin{array}{cccccc} & & & 1 & 1 & 1 \\ & 1 & 2 & 2 & & \\ 2 & 3 & 3 & & & \end{array} \qquad \begin{array}{cccccc} & & & 1 & 1 & 1 \\ & 2 & 2 & 2 & & \\ 1 & 3 & 3 & & & \end{array}$$

($c^{(6,4,3)}_{(3,1)(4,3,2)} = 2$)

Combinatorics I

Problem (Murnaghan 1938, Stanley)

Find a positive combinatorial interpretation for $g(\lambda, \mu, \nu)$, i.e. a family of combinatorial objects $\mathcal{O}_{\lambda,\mu,\nu}$, s.t. $g(\lambda, \mu, \nu) = \#\mathcal{O}_{\lambda,\mu,\nu}$.

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Theorem [Murnaghan] If $|\lambda| + |\mu| = |\nu|$ and $n > |\nu|$, then

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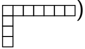
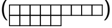
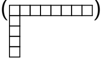
Combinatorial formulas for $g(\lambda, \mu, \nu)$, when:

- μ and ν are hooks (), [*Remmel, 1989*]
- $\nu = (n - k, k)$ () and $\lambda_1 \geq 2k - 1$, [*Ballantine–Orellana, 2006*]
- $\nu = (n - k, k)$, $\lambda = (n - r, r)$ [*Remmel–Whitehead*, 1994; *Blasiak–Mulmuley–Sohoni*,2013]
- $\nu = (n - k, 1^k)$ (), [*Blasiak 2012, Blasiak-Liu 2014*]
- Other special cases [Colmenarejo-Rosas, Ikenmeyer-Mulmuley-Walter, Pak-Panova, Mishna-Rosas-Sundaram].

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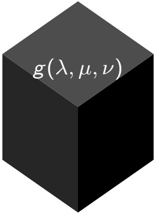
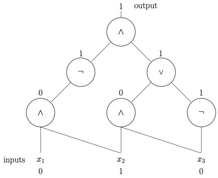


Computational Complexity in a Nutshell

`Input:` I, size$(I) = n$ (bits)
`Problem:` $C(I)$



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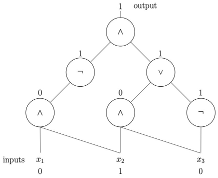

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... an object X, s.t. $X \in C(I)$?
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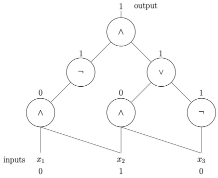

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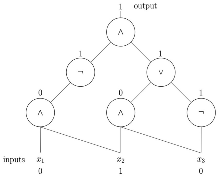

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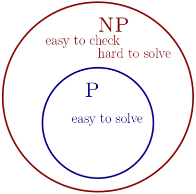

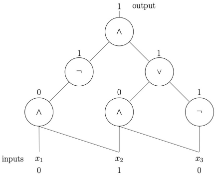

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Millennium Problem: Is P = NP **?**

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Combinatorial interpretations vs Computational Complexity

Littlewood-Richardson:

LR: Input: λ, μ, ν Output: $c^{\lambda}_{\mu\nu}$

$\mathrm{LR}\textsc{pos}$: Input: λ, μ, ν Output: Is $c^{\lambda}_{\mu\nu} > 0$?

LR rule $\Longrightarrow \mathrm{LR} \in \#\mathsf{P}$

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[Bürgisser-Ikenmeyer, Pak-P]: $\textsc{Kron} \in \mathtt{GapP}$.

[Ikenmeyer-Mulmuley-Walter]: $\textsc{KronPos}$ is [strongly] NP-hard.

Question[Pak-P]: is $\textsc{Kron} \in \#$P?

Positivity

Conjecture (Tensor square, Saxl'12)

For every $n \geq 9$ there is an irreducible S_n representations, $\mathbb{S}_\lambda$, such that $\mathbb{S}_\lambda \otimes \mathbb{S}_\lambda$ contains every irreducible representation. I.e. $g(\lambda, \lambda, \mu) > 0$ for every $\mu \vdash n$. Saxl conjecture: for $n = \binom{k}{2}$ such partition is $\lambda = \delta_k = (k-1, \ldots, 1)$ 

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Partial results:

[Pak-P-Vallejo'13]: for μ - 2-row, hook, hook + boxes etc

[PPV'13],[PP'16] $\quad g(\lambda, \lambda, \mu) \geq |\chi^\mu(2\lambda_1 - 1, 2\lambda_2 - 3, \ldots)| \quad$ for $\lambda = \lambda'$

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Other positivity results:

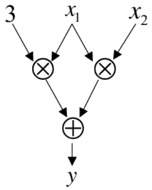
[Ikenmeyer-P,'16]:
$g((N - ab, a^b), (N - ab, a^b), (N - |\gamma|, \gamma)) > 0$ for large N and almost all γ, a, b (with some restrictions), related to Geometric Complexity Theory.

$\rightarrow$ NEXT

Algebraic P vs NP: VP vs VNP

Arithmetic Circuits:

$y = 3x_1 + x_1x_2$



Input: $X_1, \ldots, X_n$ and constants from $\mathbb{F}$.

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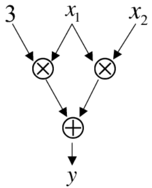

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polynomials that can be computed with circuits with $poly(n)$ nodes

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polynomials f_n, s.t. $\exists g_n \in$ VP with
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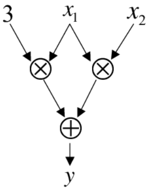

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Theorem[Bürgisser]:
If VP = VNP over finite $\mathbb{F}$ or Generalized Riemann Hypothesis holds, then P = NP.

Geometric Complexity Theory in a Nutshell

VP vs VNP : permanent vs determinant

$$\det_n := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \qquad \mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i,\sigma(i)}$$

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Conjecture [Valiant'78]:
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Obstructions λ: if $\delta_{\lambda,d,n} < \gamma_{\lambda,d,n,m}$ for $n > poly(m) \Longrightarrow$ VP $\neq$ VNP.

Geometric Complexity Theory in a Nutshell

VP vs VNP : permanent vs determinant

$$\det_n := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} x_{i,\sigma(i)} \qquad \mathrm{per}_m := \sum_{\sigma \in S_m} \prod_{i=1}^{m} x_{i,\sigma(i)}$$

Conjecture [Valiant'78]:
The (normalized) permanent $x_{11}^{n-m}\mathrm{per}_m \neq \det_n[A\mathbf{x}^T]$ ($n \times n$ determinant of affine linear forms in $\{x_{ij}\}_{i,j=1}^m$) for $n = poly(m)$. (and thus VP $\neq$ VNP)

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There exist occurrence obstructions that show $n > poly(m)$.

Theorem (Bürgisser-Ikenmeyer-P)

This Conjecture is false. There are no such occurrence obstructions for $n > m^{25}$.

Kronecker coefficients and GCT

VP vs VNP

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$$\delta_{\lambda,d,n} \leq g(\lambda, n^d, n^d) \qquad \gamma_{\lambda,d,n,m} \leq a_\lambda(d[n])$$

Kronecker coefficients and GCT

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Conjecture (GCT, Mulmuley and Sohoni)

There exist λ, s.t. $g(\lambda, n^d, n^d) = 0$ and $\gamma_{\lambda,d,n,m} > 0$ for some $n > poly(m)$.

Theorem (Ikenmeyer-P)

Let $n > 3m^4$, $\lambda \vdash nd$. If $g(\lambda, n^d, n^d) = 0$(so $mult_\lambda \mathbb{C}[GL_{n^2}\det_n] = 0$), then $mult_\lambda(\mathbb{C}[\overline{GL_{n^2}\mathrm{per}^n_m}] = 0$.

Theorem (Ikenmeyer-P)

For every partition ρ, let $n \geq |\rho|$, $d \geq 2$, $\lambda := (nd - |\rho|, \rho)$. Then $g(\lambda, n^d, n^d) \geq a_\lambda(d[n])$.

No occurrence obstructions: positive Kroneckers

Theorem (Ikenmeyer-Panova)

Let $n > 3m^4$, $\lambda \vdash nd$. *If* $g(\lambda, n \times d, n \times d) = 0$ *(so* $\mathrm{mult}_\lambda \mathbb{C}[\overline{GL_{n^2}\det_n}]_d = 0$*), then* $\mathrm{mult}_\lambda(\mathbb{C}[\overline{GL_{n^2} per_m^n}]_d = 0$.

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Proof ingredients:

Theorem (Kadish-Landsberg)

If $\mathrm{mult}_\lambda \mathbb{C}[\overline{GL_{n^2} per_m^n}]_d > 0$, then $\lambda_1 \geq nd - md$ and $\ell(\lambda) \leq m^2$.

Theorem (Degree lower bound, [IP])

If $\lambda_1 \geq nd - md$ with $\gamma_{\lambda,d,n,m} > g(\lambda, n \times d, n \times d)$, then $d > \frac{n}{m}$.

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Theorem (Kronecker positivity, [IP])

If $\ell(\lambda) \leq m^2$, $\lambda_1 \geq nd - md$, $d > 3m^3$, and $n > 3m^4$, then $g(\lambda, n \times d, n \times d) > 0$, except for 6 special cases.

Proof uses semigroup property, symmetries, positivity for squares.

Multiplicity obstructions in GCT

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Other models: Matrix power vs permanent, Iterated Matrix Multiplication vs permanent. (multiplicities for the orbits express in terms of LR, Kron, plethysms)

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Toy problem: Factor $\ell_1^n + \cdots + \ell_k^n$ into linear forms? $(k > 2)$

$$\mathrm{Ch}_m^n := \{\ell_1 \cdots \ell_n \mid \ell_i \in V\} \qquad \text{vs} \qquad \mathrm{Ps}_{m,k}^n := \overline{\{\ell_1^n + \cdots + \ell_k^n \mid \ell_i \in V\}},$$

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Theorem (Dörfler–Ikenmeyer-P'20)

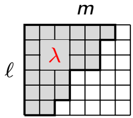
Let $m \geq 3$, $n \geq 2$. We have $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}^n_m]_{n+1}) < \mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ps}^n_{m,n+1}]_{n+1})$ for $\lambda = (n^2-2, n, 2)$, i.e., λ **is a multiplicity obstruction** *that shows $P^n_{m,n+1} \not\subseteq \mathrm{Ch}^n_m$.*
No occurrence obstructions, *for explicit values of k, n, m.*

[BIP'16] $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ps}^n_{m,k}]_d) = a_\lambda(d[n])$ for $k \geq d$.
[Landsberg] $\mathrm{mult}_\lambda(\mathbb{C}[\mathrm{Ch}^n_m]_d) \leq a_\lambda(n[d])$
Explicit plethysm formula: $a_{(n^2-2,n,2)}((n+1)[n]) = 1 + a_{(n^2-2,n,2)}(n[n+1])$

Tight asymptotics

$p_n(\ell, m) := \#\{\lambda \vdash n;\ \lambda \subset (m^\ell)\}$

$$\sum_{k \geq 0} p_n(\ell, m) q^n \;=\; \begin{bmatrix} m+\ell \\ m \end{bmatrix}_q$$



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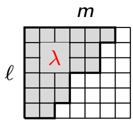

Theorem (Pak-P'15)

For all $m \geq \ell \geq 8$ *and* $2 \leq k \leq \ell m/2$, *let* $s = \min\{2k, \ell^2\}$. *We have:*

$$g(m^\ell, m^\ell, (m\ell - k, k)) = p_k(\ell, m) \, - \, p_{k-1}(\ell, m) \, > \, 0.004 \, \frac{2^{\sqrt{s}}}{s^{9/4}} \, .$$

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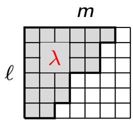

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Theorem (Melczer-P-Pemantle'19)

Let $A := \frac{\ell}{m}$ $B := \frac{n-1}{m^2}$. *Let* c, d *be solutions of [a system of integral equations]*

$$p_n(\ell, m) - p_{n-1}(\ell, m) \sim \frac{d}{m} p_{n-1}(\ell, m) \sim \frac{d \ e^{m\left[cA + 2dB - \log(1 - e^{-c-d})\right]}}{2\pi m^3 \sqrt{D}}.$$

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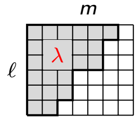

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Maximal multiplicities

Theorem [Stanley]

$$\max_{\lambda \vdash n} \max_{\mu \vdash n} \max_{\nu \vdash n} \; g(\lambda,\mu,\nu) \;=\; \sqrt{n!}\, e^{-O(\sqrt{n})},$$

$$\max_{0 \le k \le n} \max_{\lambda \vdash n} \max_{\mu \vdash k} \max_{\nu \vdash n-k} \; c^{\lambda}_{\mu,\nu} \;=\; 2^{n/2 - O(\sqrt{n})}.$$

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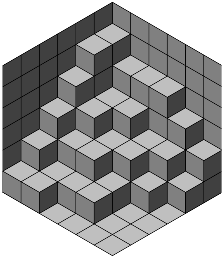
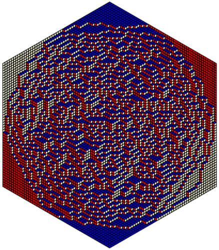
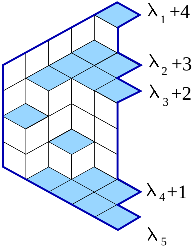
Question: [Stanley] For which λ,μ,ν are these maxima achieved?

Stat mech motivation: lozenge tilings







$$\lim_{n \to \infty} \frac{s_{\lambda^n}(x_1, \ldots, x_k, 1^{n-k})}{s_{\lambda^n}(1^n)}$$

[Gorin-P'15] effective asymptotics giving GUE near boundary, also in [Novak, Petrov] etc, subsequently used for LLN and CLT for trapezoidal domains [Bufetov-Gorin, Aggarwal-Gorin] etc

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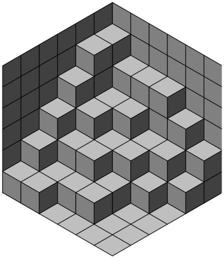

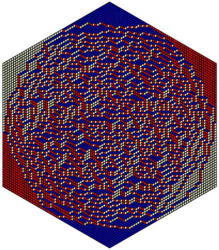

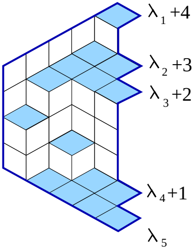

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Question: What about nontrapezoidal domains, can we analyze asymptotically $\dfrac{s_{\lambda/\mu}(x_1,\ldots,x_k,1^{n-k})}{s_{\lambda/\mu}(1^n)}$?

Question: Asymptotics of $K_{\lambda/\mu,\nu}, c^{\lambda}_{\mu\nu}$ etc as λ,μ,ν grow..?

Largest Kroneckers

Inequalities

$$\sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)^2 = \sum_{\alpha\vdash n} z_\alpha \geq z_{1^n} = n!,$$

where $z_\alpha = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$ when $\alpha = (1^{m_1} 2^{m_2} \ldots)$,

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Theorem (Pak-Panova-Yeliussizov'18)

Let $\{\lambda^{(n)} \vdash n\}$, $\{\mu^{(n)} \vdash n\}$, $\{\nu^{(n)} \vdash n\}$ be three partition sequences, such that

$$(*) \qquad g\left(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}\right) = \sqrt{n!}\, e^{-O(\sqrt{n})}.$$

Then $\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}$ are Plancherel (i.e. VKLS shape). Conversely, for every two Plancherel sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash n\}$, there exists a Plancherel partition sequence $\{\nu^{(n)} \vdash n\}$, s.t. (*) holds.

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$$\mathbf{D}(n) := \max_{\lambda\vdash n} f^\lambda$$

Theorem[PPY]: Let $\mu, \nu \vdash n$, s.t. $f^\mu, f^\nu \geq \mathbf{D}(n)/a$ for some $a \geq 1$. Then there exist $\lambda \vdash n$, s.t.

$$f^\lambda \geq \frac{\mathbf{D}(n)}{a\sqrt{p(n)}} \quad \text{and} \quad g(\lambda,\mu,\nu) \geq \frac{\mathbf{D}(n)}{a^2\, p(n)}.$$

Littlewood-Richardson

Theorem (PPY'18)

There exists a constant $d > 0$, s.t. for all $n > k \geq 1$:

$$\sqrt{\binom{n}{k}}\, e^{-d\sqrt{n}} \leq \max_{\lambda \vdash n} \max_{\mu \vdash k} \max_{\nu \vdash n-k} c^{\lambda}_{\mu,\nu} \leq \sqrt{\binom{n}{k}}.$$

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Fix $0 < \theta < 1$ and let $k_n := \lfloor \theta n \rfloor$. Then:

1. for every Plancherel partition sequence $\{\lambda^{(n)} \vdash n\}$, there exist Plancherel partition sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, s.t.

$$(**) \qquad c^{\lambda^{(n)}}_{\mu^{(n)},\,\nu^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(\sqrt{n})},$$

2. for all Plancherel partition sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, there exists a Plancherel partition sequence $\{\lambda^{(n)} \vdash n\}$, s.t. $(**)$ holds,
3. for all Plancherel partition sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash k_n\}$, there exists a partition sequence $\{\nu^{(n)} \vdash n - k_n\}$, s.t.

$$f^{\nu^{(n)}} = \sqrt{n!}\, e^{-O(n^{2/3}\log n)} \quad \text{and} \quad c^{\lambda^{(n)}}_{\mu^{(n)},\,\nu^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(n^{2/3}\log n)}.$$

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There exists a constant $d > 0$, s.t. for all $n > k \geq 1$:

$$\sqrt{\binom{n}{k}}\, e^{-d\sqrt{n}} \leq \max_{\lambda\vdash n} \max_{\mu\vdash k} \max_{\nu\vdash n-k} c^{\lambda}_{\mu,\nu} \leq \sqrt{\binom{n}{k}}.$$

Theorem (PPY'18)

Fix $0 < \theta < 1$ and let $k_n := \lfloor \theta n \rfloor$. Then:

1. for every Plancherel partition sequence $\{\lambda^{(n)} \vdash n\}$, there exist Plancherel partition sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, s.t.

$$(**) \qquad c^{\lambda^{(n)}}_{\mu^{(n)},\, \nu^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(\sqrt{n})}\,,$$

2. for all Plancherel partition sequences $\{\mu^{(n)} \vdash k_n\}$ and $\{\nu^{(n)} \vdash n - k_n\}$, there exists a Plancherel partition sequence $\{\lambda^{(n)} \vdash n\}$, s.t. $(**)$ holds,
3. for all Plancherel partition sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash k_n\}$, there exists a partition sequence $\{\nu^{(n)} \vdash n - k_n\}$, s.t.

$$f^{\nu^{(n)}} = \sqrt{n!}\, e^{-O(n^{2/3}\log n)} \quad \text{and} \quad c^{\lambda^{(n)}}_{\mu^{(n)},\, \nu^{(n)}} = \binom{n}{k_n}^{1/2} e^{-O(n^{2/3}\log n)}\,.$$

[Belinschi-Guionnet-Huang'20+]: General upper bounds on $c^{\lambda}_{\mu\nu}$ for "nice measures" via elliptical [random matrix] integrals.

Small number of rows

Theorem (Pak-P'20)

Let $\lambda, \mu, \nu \vdash n$ such that $\ell(\lambda) = \ell$, $\ell(\mu) = m$, and $\ell(\nu) = r$. Then:

$$g(\lambda, \mu, \nu) \le \left(1 + \frac{\ell m r}{n}\right)^n \left(1 + \frac{n}{\ell m r}\right)^{\ell m r}.$$

Corollary: Let $\lambda = (\ell^2)^\ell$, where $\ell = \sqrt[3]{n}$, then

$$g(\lambda, \lambda, \lambda) \le 4^n.$$

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Proof via contingency arrays:

$$T(\lambda, \mu, \nu) = \#\{(X_{i,j,k}) \in \mathbb{Z}_{\ge 0}^{\ell m r} : \sum_{j=1,k=1}^{m,r} X_{i,j,k} = \lambda_i, \sum_{i=1,k=1}^{\ell,r} X_{i,j,k} = \mu_j, \sum_{i=1,j=1}^{\ell,m} X_{i,j,k} = \nu_k\},$$

$$\sum_{\lambda,\mu,\nu} g(\lambda, \mu, \nu)\, s_\lambda(x)\, s_\mu(y)\, s_\nu(z) = \sum_{\alpha,\beta,\gamma} T(\alpha, \beta, \gamma)\, x^\alpha\, y^\beta\, z^\gamma.$$

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[Barvinok]: The number of 3d contingency tables with marginals (α, β, γ) is

$$\leq \exp\left(\max_{Z \in P(\alpha,\beta,\gamma)} \sum_{i,j,k} (Z_{ijk} + 1)\log(Z_{ijk} + 1) - Z_{ijk}\log(Z_{ijk})\right)$$

The elusive lower bound

[Bessenrodt-Behns] : $g(\lambda, \lambda, \lambda) \geq 1$ for $\lambda = \lambda'$

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Theorem [Manivel, Vallejo]

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Conjecture [Pak-P'20]:

$$\sum_{\lambda\vdash n, \lambda=\lambda'} g(\lambda,\lambda,\lambda) = \exp\left(\frac{1}{2} n\log n + O(n)\right).$$

The reduced (but not simpler!) Kronecker coefficients

$$\overline{g}(\alpha,\beta,\gamma) := \lim_{n\to\infty} g\big(\alpha[n],\beta[n],\gamma[n]\big), \qquad \alpha[n] := (n-|\alpha|,\alpha_1,\alpha_2,\ldots),\ n \geq |\alpha|+\alpha_1,$$

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Conjecture (Kirillov, Klyachko)

The reduced Kronecker coefficients satisfy the saturation property:

$$\overline{g}(N\alpha,N\beta,N\gamma) > 0 \quad \textit{for some } N \geq 1 \quad \Longrightarrow \quad \overline{g}(\alpha,\beta,\gamma) > 0\,.$$

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Theorem (Pak-P, '20)

For all $k \geq 3$, the triple of partitions $\big(1^{k^2-1}, 1^{k^2-1}, k^{k-1}\big)$ is a counterexample to the Conjecture. For every partition γ s.t. $\gamma_2 \geq 3$, there are infinitely many pairs $(a,b) \in \mathbb{N}^2$ s.t. (a^b, a^b, γ) is a counterexample.

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Example:
$\overline{g}(1^5,1^5,(3,3))=0$, but $\overline{g}(2^5,2^5,(6,6))>0$.

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$$\max_{a+b+c\leq 3n}\ \max_{\alpha\vdash a}\ \max_{\beta\vdash b}\ \max_{\gamma\vdash c}\ \overline{g}(\alpha,\beta,\gamma) = \sqrt{n!}\ e^{O(n)}$$

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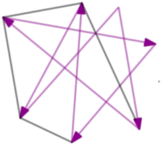
Theorem (Pak-P'20)

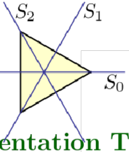
Computing the reduced Kronecker coefficients $\overline{g}(\alpha,\beta,\gamma)$ is strongly #P-hard.

Thank you!

Complexity Theory

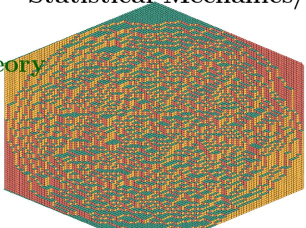
P vs NP





Representation Theory

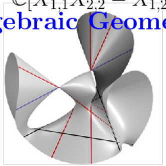
Statistical Mechanics/

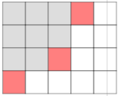


Probability

$\mathbb{C}[X_{1,1}X_{2,2} - X_{1,2}X_{2,1}]$

Algebraic Geometry





Algebraic Combinatorics

$$s_{(2,2)}(x_1,x_2,x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}}{x_1^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}}{x_2^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}}{x_1^2x_2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}}{x_1x_2^2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}}{x_1x_2x_3^2}$$