Ten questions about Hessenberg varieties

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Outline

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Definitions

The flag variety is G/B

If $G = GL_n(\mathbb{C})$ and B is upper-triangular matrices then each flag is

- … a coset gB
- … a nested subspace $V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{C}^n$

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$$\left\langle \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \subseteq \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

- … a matrix with zeros to the right and below a permutation

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Definitions

Each gB has a representative in exactly one of the following:

$$\begin{pmatrix} * & * & 1 \\ * & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

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These are the **Schubert cells** BwB/B. They are parametrized by permutation matrices w.

Definitions: Springer fibers

Fix a linear operator $X : \mathbb{C}^n \rightarrow \mathbb{C}^n$

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For example: The Springer fiber of $X = 0$ is the full flag variety.

Defitions: Springer theory

Theorem

Suppose X is nilpotent and $\mathcal{S}_X$ is the Springer fiber of X.

- *S_n acts naturally on the cohomology $H^*(\mathcal{S}_X)$*
- *The top-dimensional cohomology of $H^*(\mathcal{S}_X)$ is irreducible of type $\lambda(X)$*
- *The set $\{H^{top}(\mathcal{S}_\lambda)\}$ is precisely the collection of irreducible representations of S_n when λ ranges over nilpotent conjugacy classes (or partitions of n)*

Springer '76, Kazhdan-Lusztig '80, Borho-MacPherson '83, Hotta '82, Lusztig '84, Garsia-Procesi '92, and others...

Definitions: the parameter h (or H)

A *Hessenberg function* $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a map that is both

- nondecreasing, meaning $h(i) \geq h(i-1)$ for all i
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$$h = (2, 3, 4, 4) \qquad \longleftrightarrow \qquad H = \begin{array}{|c|c|c|c|} \hline * & * & * & * \\ \hline * & * & * & * \\ \hline 0 & * & * & * \\ \hline 0 & 0 & * & * \\ \hline \end{array}$$

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For example: The Hessenberg variety of $h = (n, n, \ldots, n)$ is the full flag variety for all X.

Definitions: Hessenberg varieties

Examples of Hessenberg varieties:

- Springer fibers: when X is nilpotent and $h(i) = i$ for all i
- Permutation flags: when X is regular semisimple and $h(i) = i$ for all i
- The toric variety associated to the permutohedron: when X is regular semisimple and $h(i) = i + 1$ for all $i < n$
- The Peterson variety: when X is principal nilpotent and $h(i) = i + 1$ for all $i < n$

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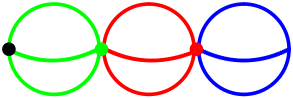
Theorem (T–, Precup)

If a basis for $\mathbb{C}^n$ is chosen correctly (equivalently if X is chosen in the correct relative position to B) then the Schubert cells pave $\mathcal{H}ess(X, H)$ by affines.



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Hessenberg varieties combine the combinatorics of permutations (via Schubert cells) with the combinatorics of nilpotent orbits (via the containment condition).

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Example: X has one Jordan block, $h(1) = 2, h(2) = 3, h(3) = 3$.

$$X \begin{pmatrix} a & b & 1 \\ c & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For this matrix to be in the Hessenberg variety

$$\begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix} \in span \left\langle \begin{pmatrix} a \\ c \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ 1 \\ 0 \end{pmatrix} \right\rangle$$

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$$\begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix} \in span \left\langle \begin{pmatrix} a \\ c \\ 1 \end{pmatrix}, \begin{pmatrix} b \\ 1 \\ 0 \end{pmatrix} \right\rangle \qquad \text{so } b = c$$

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A *paving by affines* is less restrictive than a CW decomposition.

- A paving by affines is an ordered partition into cells homeomorphic to $\mathbb{C}^{d_i}$ so that the boundary of the i^{th} cell is contained in the union of cells $1, 2, \ldots, i-1$

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- Boundaries may contain *pieces* of cells of the same dimension
- Cell closures in a paving by affines induce a (co)homology basis

Question 1: Structure constants

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Let $p_w = [\overline{\mathcal{C}_w \cap \mathcal{H}ess(X,h)}]$ be the cohomology class. What are the coefficients c_w^{uv} in

$$p_u p_v = \sum_{w \in S_n} c_w^{uv} p_w$$

Question 2: Closures of Hessenberg Schubert cells

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- The Schubert variety $\overline{BwB/B}$ is smooth if and only if the permutation w avoids certain patterns[2]
- The type of singularity in $\overline{BwB/B}$ is determined by the kind of pattern w contains, more generally combinatorics of w[3]

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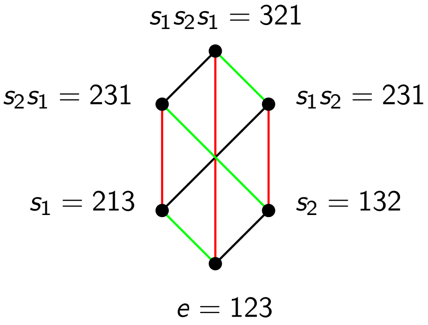
[1] Chevalley; [2] Lakshmibai-Sandhya; [3] Kumar, Carrell-Kuttler, Woo-Yong

Question 2: Bruhat order



Question 2: 2-row Springer fibers

Theorem

Springer fibers are paved by affine cells $C_w \cap \mathcal{S}_X$. The cells $C_w \cap \mathcal{S}_X$ that are nonempty are enumerated by row-strict tableaux of shape $\lambda(X)$.

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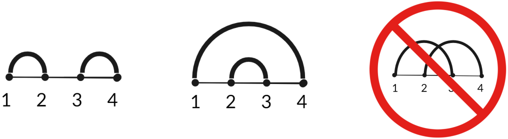

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Theorem (Goldwasser, Sun, T)

Consider the (n, n) Springer fibers, namely with two Jordan blocks of the same size. We give an explicit bijection between cells and (dotted) noncrossing matchings, in which entries in the cell correspond to arcs. The closures of the cells are given by nesting/unnesting.

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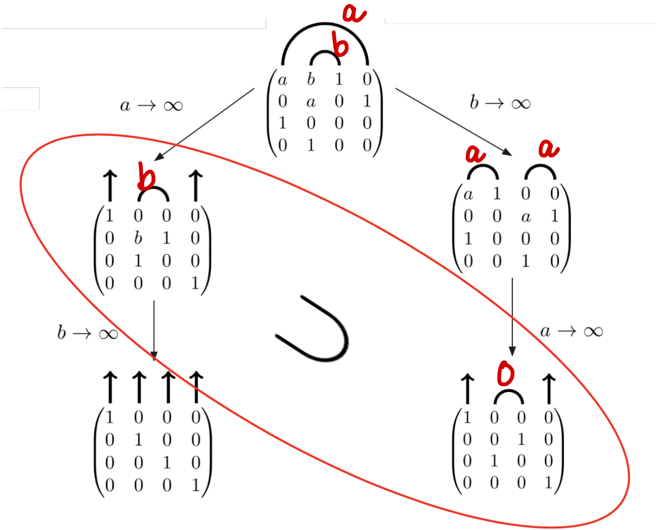
We have some work generalizing this to (n, n, n) Springer fibers, but it gets very complicated very quickly.

Question 2: 2-row Springer fibers



Question 3: Singularities

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Subquestion: when are all components of $\mathcal{H}ess(X, h)$ smooth?

- Fresse and Melnikov: Springer fibers
- Fung, Khovanov: 2-row Springer fibers
- Kostant: Peterson varieties
- Insko and Yong: Peterson varieties
- Abe, DeDieu, Galetto, Harada: regular nilpotent Hessenbergs
- Insko and Precup: semisimple Hessenberg varieties

Question 4: Components

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Variations of this question:

- When do we have a concise formula for the dimension or number of components of Hessenberg varieties? If X is regular nilpotent, the Poincare polynomial of $\mathcal{H}ess(X, h)$ is

$$\prod_{i=1}^{n}(1 + t + t^2 + \cdots + t^{h(i)-i})$$

- When is a Hessenberg variety Schubert, and vice versa? See Escobar, Precup, Shareshian for A_n, C_n pattern-avoidance

Question 5: Intersections

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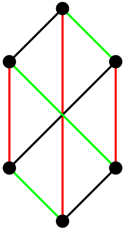
Let $\mathcal{X}_u^h = \overline{\mathcal{C}_u \cap \mathcal{H}ess(X,h)}$ and $\mathcal{X}_v^{h,opp} = \overline{B^- uB/B \cap \mathcal{H}ess(X,h)}$.

What is the intersection

$$\mathcal{X}_u^h \cap \mathcal{X}_v^{h,opp}$$

Question 5: Bruhat order and intervals



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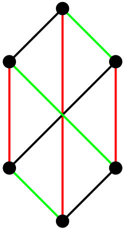

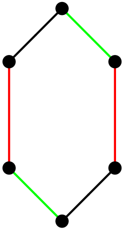

Question 6: Other pavings

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What other interesting pavings do Hessenberg varieties have?

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Even pavings by other Schubert cells can be interesting!

- Peterson: paved regular nilpotent Hessenberg varieties by Schubert cells using the opposite Borel.
- Kostant: the coordinate ring of the intersection of the big (opposite) cell with the Peterson variety is the quantum cohomology of the flag variety.
- Rietsch: the quantum parameters are minors of the matrix representatives of this intersection.

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- Rietsch: the quantum parameters are minors of the matrix representatives of this intersection.
- Precup's paving uses different Schubert cells than T
- Fresse: different paving(s) for Springer fibers
- Adeyemo and Olasupo: compare Fresse and T
- DeWitt and Harada: fillings for permuted Schubert cells

Question 7: Torus actions

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Which tori act on each Hessenberg variety?

Question 7: Torus actions

Tori are one of the ways we create pavings and identify permutations in closures.

The torus T of diagonal matrices acts by multiplication:

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \begin{pmatrix} a & b & 1 \\ c & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} t_1 a & t_1 b & t_1 \\ t_2 c & t_2 & 0 \\ t_3 & 0 & 0 \end{pmatrix} B$$

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...but the entries of Hessenberg Schubert cells aren't free.

Question 7: Torus actions

- De Mari, Procesi, and Shayman: If X is semisimple then the full torus acts. If X is regular semisimple and $h(i) = i + 1$ then $\mathcal{H}ess(X, h)$ is the toric variety associated to the decomposition into Weyl chambers.
- Harada and T: generic S^1-action
- Goldin and T: various X and h in which larger tori act
- Abe and Crooks: the full torus acts when X is in the minimal nonzero nilpotent orbit, e.g. $X = E_{1n}$

Question 8: Equivariant cohomology

Question 8:

With respect to which tori are Hessenberg varieties GKM, and what is their equivariant cohomology?

Question 8: GKM theory

Suppose $\mathcal{Y}$ is a complex projective variety with the action of a torus T.

Functoriality: The injection $\mathcal{Y}^T \hookrightarrow \mathcal{Y}$ induces a map

$$H_T^*(\mathcal{Y}) \to H_T^*(\mathcal{Y}^T)$$

Kirwan, Atiyah, ...: For some ("equivariantly formal") X

$$H_T^*(\mathcal{Y}) \hookrightarrow H_T^*(\mathcal{Y}^T)$$

Chang-Skjelbred: We can identify the image of this map inside

$$H_T^*(\mathcal{Y}^T) \cong \bigoplus_{v \in \mathcal{Y}^T} \mathbb{C}[t_1, \ldots, t_n]$$

Question 8: Moment graphs

Definition

The moment graph $\Gamma_{\mathcal{Y}}$ of $\mathcal{Y}$ is an edge-labeled graph for which:

- vertices are $\mathcal{Y}^T$,
- edges are pairs u, v that are in the closure of a particular 1-dimensional T-orbit, and
- for each edge e, the label α_e is the T-weight on the corresponding 1-dimensional orbit.

The *moment graph* is the 0- and 1-skeleton of the moment polytope.

Question 8: GKM theorem

Goresky-Kottwitz-MacPherson identify conditions on X so that we can compute the image of $H^*_T(\mathcal{Y}) \to H^*_T(\mathcal{Y}^T)$ combinatorially.

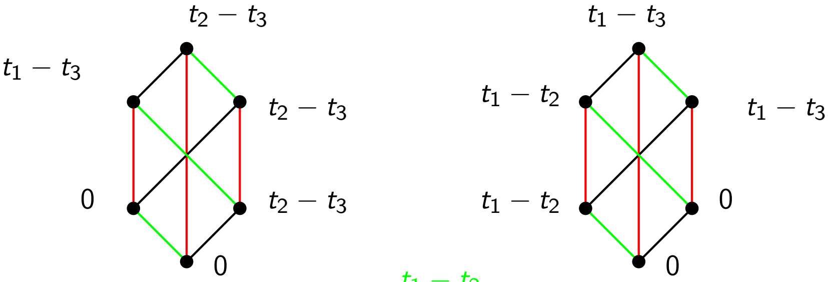
Theorem

(GKM) If $\mathcal{Y}$ is equivariantly formal, $\mathcal{Y}^T$ is finite, and $\mathcal{Y}$ has finitely many 1-dimensional orbits, then

$$H^*_T(\mathcal{Y}) \cong \left\{ p \in \mathbb{C}[t_1, \ldots, t_n]^{|X^T|} : \begin{array}{c} \text{for each edge } e \text{ in } \Gamma_X, \\ p_v - p_u \in \langle \alpha_e \rangle \end{array} \right\}$$

Examples: flag varieties, Grassmannians, G/B, G/P

Question 8: Testing ring elements



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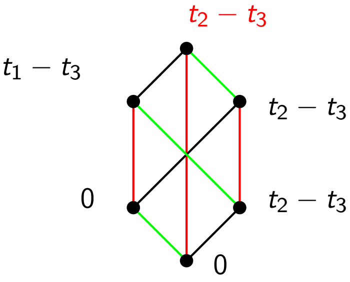
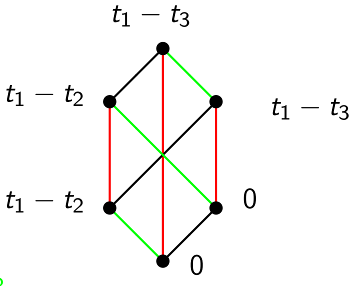


NO



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Question 8: Some GKM Hessenberg varieties

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- Goldin and Tymoczko: fully classify $X = E_{1,n-1} + E_{2,n}$ plus other cases
- Argaez and Zaldivar: semisimple

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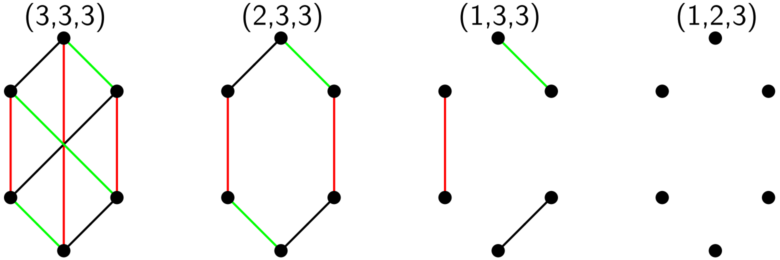
- Argaez and Zaldivar: semisimple
- GKM-ish: Drellich: Constructs Monk and Giambelli formulas for S^1-equivariant cohomology of Peterson varieties
- Abe, Harada, Horiguchi, and Masuda: regular nilpotent
- Argaez and Zaldivar: regular nilpotent

Question 8: Moment graphs if X regular semisimple

- Vertices are permutations.
- Edges are pairs $(ij)w \leftrightarrow w$ with $w^{-1}(i) \leq h(w^{-1}(j))$.



Question 9: S_n representations

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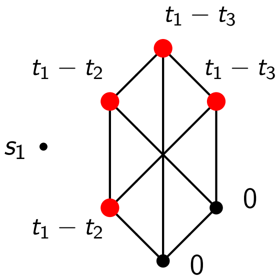
- Or: what is the largest subgroup of S_n that acts on the cohomology in an interesting way?
- And: what is the representation, in the sense of an explicit combinatorial decomposition into irreducibles (or other favorite basis).

Question 9: The dot action

The action of s_i on a GKM class

- acts on polynomial vertex-label and
- *exchanges* vertex labels over edges labeled $t_i - t_{i+1}$



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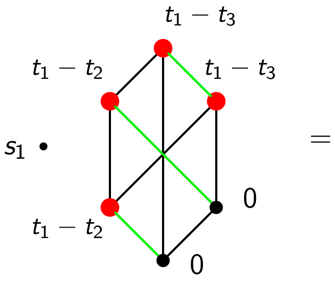

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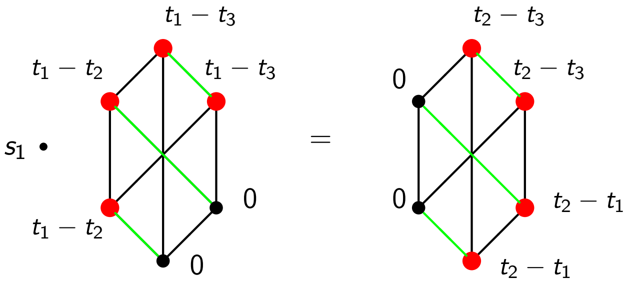

Question 9: The Stanley-Stembridge conjecture

Stanley and Stembridge's (3+1)-conjecture concerns the chromatic symmetric function of the incomparability graph of a kind of poset called *(3+1)-free*. The (3+1)-conjecture is that this symmetric function is a nonnegative linear combination of elementary symmetric functions.

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The Stanley-Stembridge conjecture would be proven if we found a basis for the GKM cohomology of the regular semisimple Hessenberg variety that the S_n-action permutes.

Question 10: Beyond type A_n

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Which of these results holds for other Lie types?

Question 10: Beyond type A_n

- Precup: general type, some X
- Drellich: Monk and Giambelli for Peterson varieties of classical type
- Abe, Horiguchi, Masuda, Murai, Sato: presentation of $\mathcal{H}ess(X, H)$ when X regular nilpotent in types B, C, G using hyperplane arrangements
- Springer theory

THANK YOU!

Let λ is a partition of n, e.g. dimensions of the Jordan blocks of X.

Definition

The Young diagram of shape λ is an arrangement of boxes (left- and top-aligned) with λ_i boxes in the i^{th} row. A Young tableau is a Young diagram that has been filled with numbers according to some rule:

- **row-strict** means the numbers increase in each row (L to R)
- **standard** means row-strict and that numbers increase in each column (top to bottom)

1	3
2	4
5	

2	4
1	3
5	

Question 1

Theorem (Tymoczko)

Let X be nilpotent and λ be the Young diagram for X. Schubert cells with nonempty $\mathcal{C}_w \cap \mathcal{H}ess(X, h)$ correspond to Young tableaux with

$$i \leq h(j) \text{ for each adjacent pair } \boxed{i \mid j}$$

The dimension of $\mathcal{C}_w \cap \mathcal{H}ess(X, h) \neq \emptyset$ is the number of pairs i, k with

- *k to the left of or in the same column and below i*
- *$i < k$ and*
- *either i is rightmost in its row or $\boxed{i \mid j}$ and $k \leq h(j)$*

Fillings tell you when to add each basis vector to flag

$$i \leq h(j) \text{ for each adjacent pair } \boxed{i \mid j}$$

- Springer fiber: When $h(i) = i$ for all i then each Young tableau satisfies

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....row is partitioned into blocks $j, j-1, j-2, \ldots, j-k$

1	2	3

2	1	3

1	3	2

3	2	1

Dimension counts certain inversions: the number of pairs i, k with

- k to the left of or in the same column and below i
- $i < k$ and
- either i is rightmost in its row or $\boxed{i \mid j}$ and $k \leq h(j)$

III. Four applications: banded Hessenberg form

Definition

A matrix is in *banded Hessenberg form* if it is zero below the subdiagonal.

$*$	$*$	$*$	$*$
$*$	$*$	$*$	$*$
0	$*$	$*$	$*$
0	0	$*$	$*$

- The *QR algorithm* to find eigenvalues and eigenvectors is more efficient on matrices in banded Hessenberg form.
- Suppose $h(i) = i + 1$ for all $i < n$. The Hessenberg variety of X and h is the collection of ordered bases that put X into banded Hessenberg form.

III. Four applications: Springer theory

Theorem

Suppose X is nilpotent and $h(i) = i$ for all i.

- *S_n acts naturally on the cohomology $H^*(\mathcal{S}_X)$*
- *The top-dimensional cohomology of $H^*(\mathcal{S}_X)$ is irreducible of type $\lambda(X)$*
- *The set $\{H^{top}(\mathcal{S}_\lambda)\}$ is precisely the collection of irreducible representations of S_n when λ ranges over nilpotent conjugacy classes (or partitions of n)*

Springer '76, Kazhdan-Lusztig '80, Borho-MacPherson '83, Hotta '82, Lusztig '84, Garsia-Procesi '92, and others...

III. Four applications: Springer theory

The Kazhdan-Lusztig polynomials are the entries of the change-of-basis matrix between the Schubert basis for the Springer representation and the so-called canonical basis defined by Kazhdan-Lusztig.

III. Four applications: Webs

An $\mathfrak{sl}_k$ web is a plane graph that is a morphism in a diagrammatic category encoding the representation theory of $U_q(\mathfrak{sl}_k)$.

Webs satisfy skein-theoretic relations arising from the algebra they encode. We consider the vector space they generate as formal vectors, up to the equivalence relations.

When $k = 2$: Kuperberg describes the combinatorial representation theory of webs (including for other Lie types). Khovanov uses them to construct the cohomology of certain Springer fibers.

III. Four applications: Webs

For $k = 2$ the webs are noncrossing matchings or Temperley-Lieb diagrams:

III. Four applications: Webs

For $k = 3$ webs are planar, directed graphs with boundary such that the following hold:

- boundary vertices have degree one (for us, always sources)
- interior vertices are trivalent
- vertices are either sources or sinks

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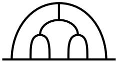

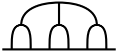

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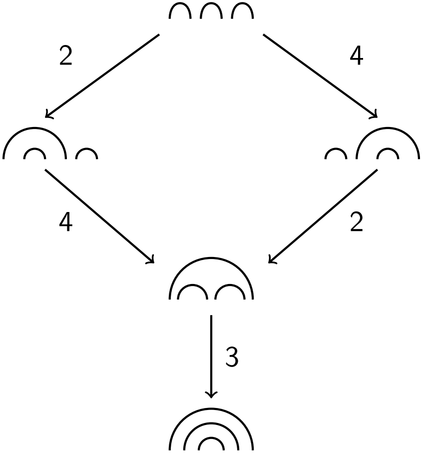
Reduced webs have no interior faces with fewer than six edges.
Reduced webs form a basis for the web vector space.

III. Four applications: Webs

S_n acts on webs by braiding their strands and then resolving according to the skein-theoretic reductions.



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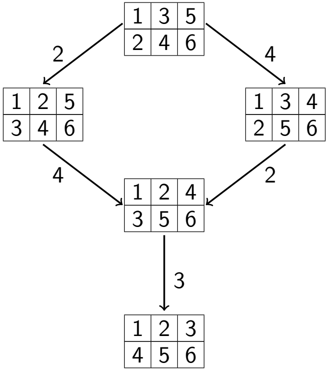

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Webs and tableaux represent two natural bases for this S_n-representation.

Theorem (Russell, T)

The natural map between webs and tableaux is not S_n-equivariant. However, the transition matrix is upper-triangular with ones along the diagonal.

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- Rhoades proved that the entries are nonnegative.
- Im and Zhu recently proved that the vanishing terms we identified are in fact the only zero entries in the transition matrix.

III. Four applications: Webs

We have some work generalizing this to webs for $\mathfrak{sl}_3$.

We believe the connections to Springer fibers can help illuminate web bases for $\mathfrak{sl}_k$ when $k \geq 4$.

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