Howard Math 156: Calculus I Fall 2024
Instructor: Sam Hopkins (sam.hopkins@howard.edu)
Call me "Sam"

8/21 Logistics:

Classes: MTWF 2:10-3pm, Douglass Hall - #26
Office hrs: Tue 1-2pm, Annex III - #220
or by appointment - email me!

Website: samuelfhopkins.com/classes/156.html

Text: Calculus, Early Transcendentals, by Stewart, 7^{th} Ed.

Grading: 35% (in person) quizzes
45% 3 (in person) midterms
20% final Exam

There will be 11 in person quizzes taken on Tuesdays (about 20 mins, we'll go over them for rest of class) Your lowest 2 scores will be dropped (so 9/11 count)

The 3 midterms will happen in class, also on Tuesdays

The final will be during finals week.

Beyond that, I may assign practice problems (not graded) and I expect you to SHOW UP TO CLASS + PARTICIPATE :)

that means... interrupt me by ASKING QUESTIONS!

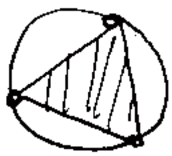
and please say your name when you ask a question, so I can start to put names to faces.

What is calculus about?

Calculus is different from the math you've seen.
It deals with change, with infinities (and infinitesimals) and with limiting processes.
It's good to have a preview of this new stuff...

Area of a circle

We all know that the area of a circle of radius R is πR^2, where $\pi = 3.14159\ldots$ is a special number.
But how would you figure this out if you didn't know?



$n=3$

You could try to approximate the area by using a simpler shape, like a regular triangle, whose area you already know how to compute.
But this clearly leaves some of the area out...
So you might consider regular 4-gon, 5-gon, ...

$n=4$ $n=5$ $n=6$

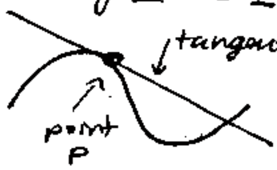
Each inscribed regular n-gon gives a better and better approximation to area of circle!

And true area can be obtained as limit as $n \to \infty$ (n goes to infinity)!

We won't study this exact problem this semester, but we will consider the area under a curve;

this can also be obtained by a limit of simpler shapes: thin rectangles under curve

Tangent to a curve: How would you find the tangent line to a curve at a point?



The tangent is the line that "just touches" the curve at that point

Calling this point P, can draw secant line through P and Q, another nearby point on curve:

. . .

As we move this other point Q closer and closer to P, the secant becomes a better and better approximation of the tangent, and in the limit, secant becomes tangent!

Why care about tangents to curves?
They tell us about velocity and acceleration in physics (and rates of change in sciences in general ...)

Also, can approx. curve by a series of tangents:
("Newton's method"... used by NASA)

← "linear approx. of curve"

Big idea of Calculus:
Even though the area problem and the tangent line problem seem pretty different...
They are actually the same problem!
Or more precisely, they are opposite problems!
This semester we'll learn why (+ how)!

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Functions (§1.1 of textbook)

Functions are the basic thing we will study in calculus.
They are fundamental in all sciences as models.

- E.g. If we produce x units of some product,
our revenue may be given by the function

$R(x) = p \cdot x$ where p = price per unit of product

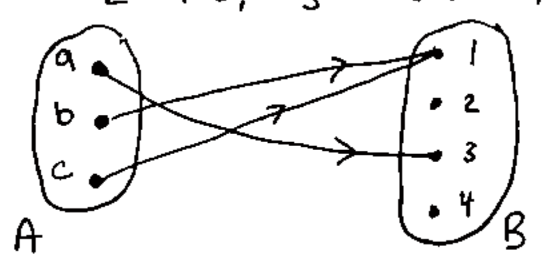
(very simple linear model, doesn't take into account costs...)

We will see derivative $R'(x)$ (slope of tangent at point x)
is what economists would call "marginal revenue."

- But what is a function?

Formally, a function f between two sets A and B
is a relation between the elements of A and B
such that every element of A is related
to exactly one element of B.

e.g. $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$



$f(a) = 3$
$f(b) = 1$
$f(c) = 1$

The set A is called the domain of f and
the set B is called the codomain of f.

The range of f is the set of all $f(x)$ for $x \in A$.

E.g. Range for f above is $\{1, 3\}$ (actual values of inputs).

The function is called <u>one</u>-<u>to</u>-<u>one</u> if every element in the range is the output of a <u>unique</u> $x \in A$.

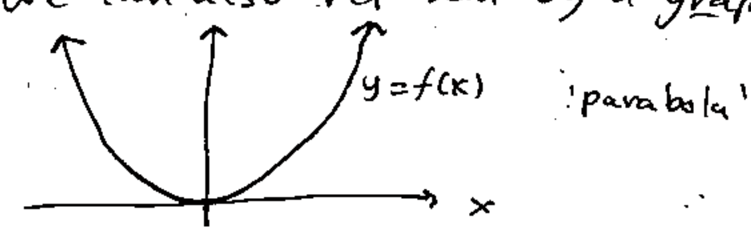
E.g. Example f above is <u>not</u> one-to-one
since $f(b) = 1$ and also $f(c) = 1$.

That is the formal definition of a function, but we will normally work with functions f whose domain & range are subsets of real numbers $\mathbb{R}$.
Then we'll have several other ways to represent f, beyond an "arrow diagram" or "chart"
(and we'll need other ways since there are ∞-many real #'s!).

You are probably used to functions defined by <u>algebraic formulas</u>, such as

$$f(x) = x^2$$

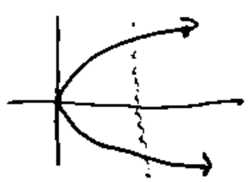
which we can also represent by a <u>graph</u>:



How do we know if a graph represents a function?

"<u>vertical line test</u>": graph represents a function $\Leftrightarrow$ each vertical line intersects ≤ 1 point on graph

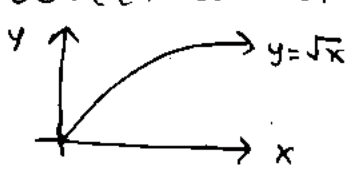
<u>E.g.</u>



graph $x = y^2$ is <u>NOT</u> graph of a function $y = f(x)$,
since vertical line $x = 4$ intersects <u>two points</u> on the graph!

The domain of $f(x) = x^2$ is all of the real numbers $\mathbb{R}$, also denoted $(-\infty, \infty)$ in "interval notation."
The range is the nonnegative reals, or $[0, \infty)$.

What about $f(x) = \sqrt{x}$?



we mean positive square root when we write this

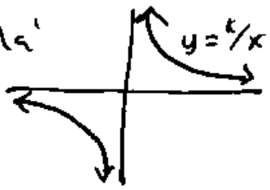
The domain is $[0, \infty)$, and range is also $[0, \infty)$.

In general, to find the domain of a function $f(x)$ think about what values you're allowed to plug into f.

E.g. Domain of $\sqrt{x-1}$ is $\{x \in \mathbb{R} : x \geq 1\} = [1, \infty)$
since can only take square root of a nonneg. #.

E.g. Function $f(x) = 1/x$ has domain (and also range)
$\{x \in \mathbb{R} : x \neq 0\} = (-\infty, 0) \cup (0, \infty)$
since we are not allowed to divide by zero.
(Denominators can never be 0.).

"hyperbola"



Can also test one-to-one-ness graphically, using:

"horizontal line test": function f is one-to-one $\iff$ every horizontal line intersects graph $y=f(x)$ in ≤ 1 point.

E.g.

$f(x) = x^2$ is Not one-to-one.

Q: what about $f(x) = x^3$?

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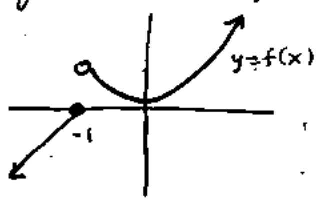
Not every function is determined by a single formula. We can define a <u>piecewise function</u> like

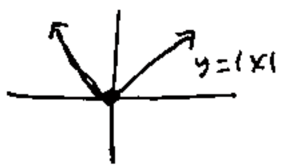
$$f(x) = \begin{cases} x+1 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

The graph of $y = f(x)$ has two parts



(see how we use ○ (empty circle) to denote a 'discontinuity')

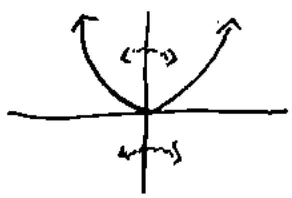
Another important piecewise function is <u>absolute value</u>:



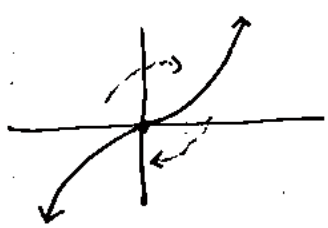
$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

← graph of $|x|$ has two parts, but they 'touch' each other

<u>Symmetries of functions</u>



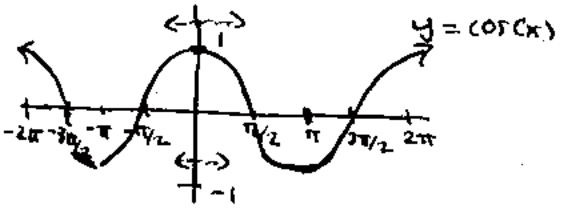
← The graph of function $f(x) = x^2$ is <u>symmetric about</u> the <u>y-axis</u>: if I <u>reflect</u> it about the y-axis (vertical axis), I get back the same thing.

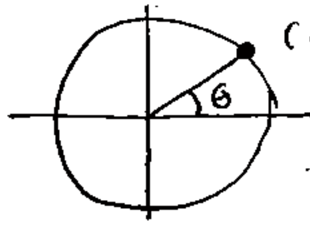


← The graph of function $f(x) = x^3$ is <u>symmetric about</u> the <u>origin</u>: if I <u>rotate</u> it 180° about origin ((0,0) point), I get back the same thing.

These two kinds of symmetry are called <u>even</u> and <u>odd</u> for functions $f(x)$.

A function $f(x)$ is called <u>even</u> if $f(x) = f(-x)$ for all x.
Same as saying graph is symmetric about y-axis.
Examples of even fn's: x^2, x^2+1, x^4, $|x|$, $\cos(x)$

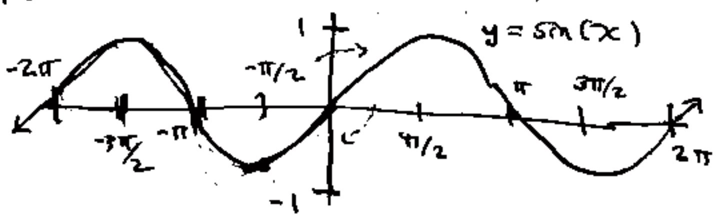




$(\cos\theta, \sin\theta)$

recall that $\cos(\theta)$ and $\sin(\theta)$ give x & y coordinates of pt on unit circle at θ <u>radians</u> ccwise from pos. x-axis

A function $f(x)$ is <u>odd</u> if $f(-x) = -f(x)$ for all x.
Same as saying graph symmetric about origin.
Examples of odd fn's: x^3, x, x^5+x^3, $\sin(x)$



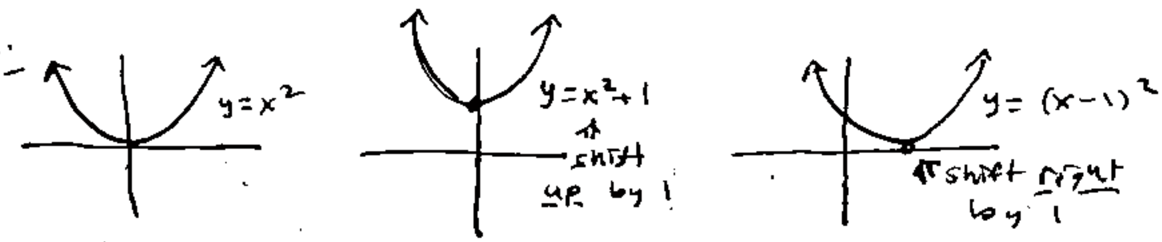
8/28 Can you guess why we use names "<u>even</u>" and "<u>odd</u>"?

§1.3 <u>Transformations of functions</u>:

Given $f(x)$ can make new functions by applying various transformations, like <u>translations</u>:

$y = f(x) + c$ - function whose graph is $f(x)$ translated <u>up</u> by c
$y = f(x) - c$ - graph is $f(x)$ translated <u>down</u> by c
$y = f(x-c)$ - graph is $f(x)$ translated <u>right</u> by c,
$y = f(x+c)$ - graph is $f(x)$ translated <u>left</u> by c
(for $c > 0$)

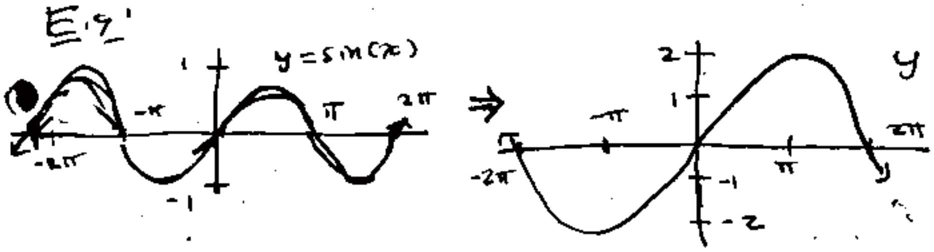
E.g.:



Can also stretch a function: for $c > 1$,

$y = c \cdot f(x)$ – stretch graph vertically by factor of c
$y = 1/c \cdot f(x)$ – shrink graph vertically by c
$y = f\left(\frac{x}{c}\right)$ – stretch graph horizontally by c
$y = f(c \cdot x)$ – shrink graph horizontally by c

E.g.:



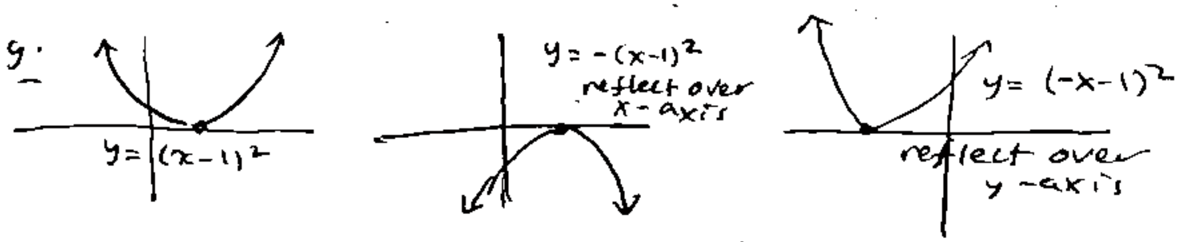
$y = 2\sin\left(\frac{1}{2}x\right)$
stretch vertically and horizontally by a factor of 2.

We see in this example how we can combine multiple transformations!

One more geometric transformation: reflection:

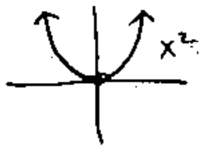
$y = -f(x)$ – reflect graph about x-axis
$y = f(-x)$ – reflect graph about y-axis

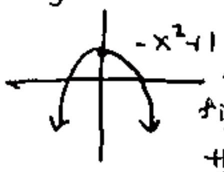
E.g.



Q: What happens w/ reflections for even + odd fn's?

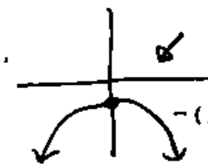
§1.3 When applying multiple transformations, order is important!





first reflect over x-axis then shift up by one

VS.



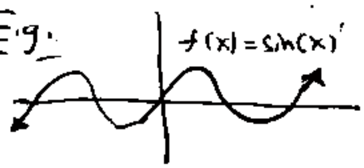
First shift up by one, then reflect over x-axis

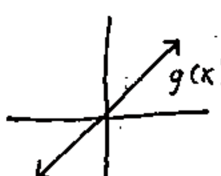
8/30 Another way to get new functions from old ones is by combining functions in various ways.

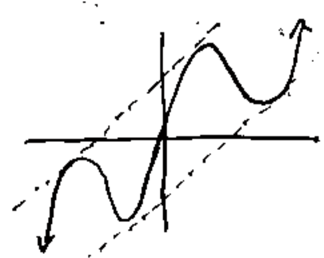
<u>Def'n</u> If f, g are two fn's, we define their <u>sum</u>, <u>difference</u>, <u>product</u>, and <u>quotient</u> by

$$(f+g)(x) = f(x)+g(x) \qquad (f-g)(x) = f(x)-g(x)$$
$$(f\cdot g)(x) = f(x)\cdot g(x) \qquad (f/g)(x) = f(x)/g(x)$$

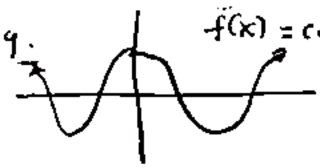
<u>E.g.</u>

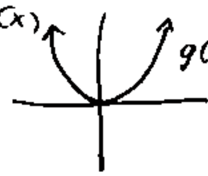


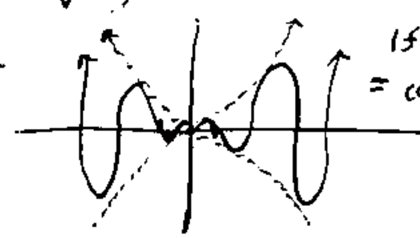




<u>E.g.</u>







<u>E.g.</u> $\tan(x) = \frac{\sin(x)}{\cos(x)}$ ← not always <u>easy</u> to graph combinations.

<u>Note</u>: Since $\cos(\frac{\pi}{2}+n\cdot\pi)=0$ for all integers $n\in\mathbb{Z}$,

the <u>domain</u> of $\tan(x) = \{x\in\mathbb{R} : x\neq\frac{\pi}{2}+n\pi \text{ for any } n\in\mathbb{Z}\}$

(because we don't want to <u>divide</u> <u>by</u> <u>zero</u>!)

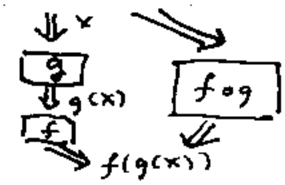
Another very important way to combine functions is <u>composition</u>:

<u>Def'n</u> If f and g are two functions, their <u>composition</u> $f \circ g$ is:

$$(f \circ g)(x) = f(g(x))$$

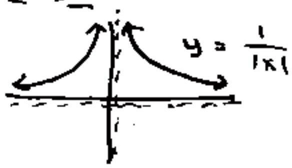
"Do g <u>first</u>, then do f to <u>that</u>!"

"f <u>of</u> g <u>of</u> x"



E.g. $f(x) = x^2$, $g(x) = 2x-1$, $(f \circ g)(x) = (2x-1)^2 = 4x^2 - 4x + 1$

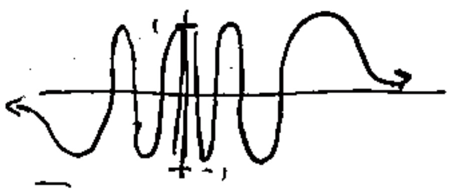
E.g. $f(x) = 1/x$, $g(x) = |x|$, $(f \circ g)(x) = \frac{1}{|x|} = \begin{cases} 1/x & \text{if } x > 0 \\ -1/x & \text{if } x < 0 \end{cases}$



<u>Note</u>: $\frac{1}{|x|}$ is <u>even</u> since $\frac{1}{|-x|} = \frac{1}{|x|}$ and $x = 0$ is <u>not</u> in the <u>domain</u>.

E.g. $f(x) = \sin(x)$, $g(x) = 1/x$, $(f \circ g)(x) = \sin(1/x)$

What does $\sin(1/x)$ look like? As $|x| \to \infty$, $1/x \approx 0$ barely changes, so $\sin(1/x)$ stops oscillating for big x. But as $|x| \to 0$, $1/x$ changes <u>a</u> <u>lot</u>, so $\sin(1/x)$ oscillates a <u>ton</u> near $x = 0$!



⇐ very hard to draw accurate graph of $y = \sin(1/x)$.

Also, <u>notice</u> $x = 0$ <u>Not</u> in <u>domain</u> (since $1/0$ not defined)

If $(f \circ g)(x) = x$, then we say that f is the <u>inverse function</u> of g. f "<u>undoes</u>" what g does!

E.g. $f(x) = \sqrt{x}$, $g(x) = x^2$, $(f \circ g)(x) = \sqrt{x^2} = x$ (for $\underline{\text{nonnegative}}$ $x \geq 0$)

The square root function $f(x) = \sqrt{x}$ "<u>undoes</u>" the square $g(x) = x^2$

We will be more careful about <u>domain</u> issues for inverses later...

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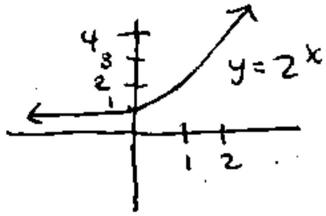
§1.4 Exponential functions

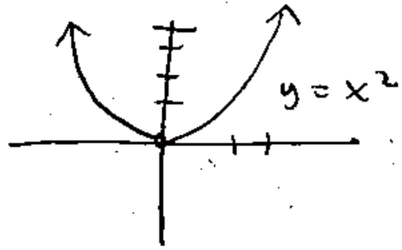
Def'n Fix real number $a > 0$. The exponential function with base a is $f(x) = a^x$.

Do not confuse a^x with power function x^a

E.g.: $f(x) = 2^x$ vs. $g(x) = x^2$

x	$f(x)$	$g(x)$
0	1	0
1	2	1
2	4	4
3	8	9
4	16	16





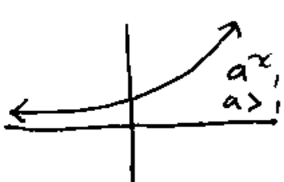
At first, x^2 grows more quickly than 2^x, but this is misleading: eventually, 2^x grows much, much faster than x^2!

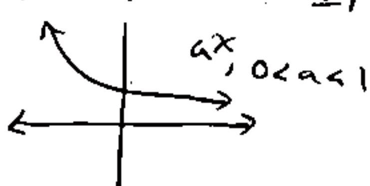
In fact, any exponential a^x for $a > 1$ (eventually) grows much, much faster than any polynomial.

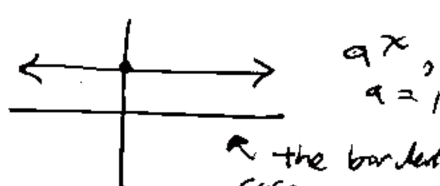
(Recall a polynomial is a function $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ that is a linear combination of power functions.)

We will prove this assertion later (using calculus!).

For $a > 1$, a^x represents exponential growth, while for $0 < a < 1$, a^x represents exponential decay:





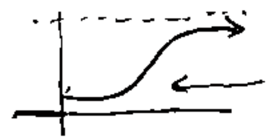


the borderline case $a = 1$ is degenerate

Sometimes we also consider Ca^x for constant C an exponential function.

In sciences, e.g. biology, often see a mix of exponential growth and decay:

e.g.



Remember: fixed exponent $x^a \Rightarrow$ power function

Fixed base $\Rightarrow$ exponential function

(So something like $f(x) = x^x$ is neither.)

The special number e: There is one base that is "best";

the number $e \approx 2.718\ldots$ ← irrational number, like π

How to define e precisely? Can use a limit:

$$e = \lim_{n\to\infty} (1 + 1/n)^n$$

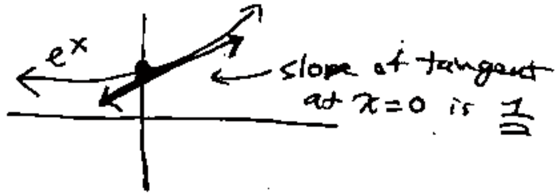
There is a way to think of this formula in terms of compound interest:

If you have \$1 invested in an investment with a rate of return of 100% per year that is "continuously compounded", then at the end of the year you will have \$$e$ $= \$2.718\ldots$

You may remember formula Pe^{rt} for interest.

Principal, rate of return, time

There is also a geometric way to think about e:



Of all exponential functions $f(x) = a^x$, the unique one that has a tangent line slope of 1 at $x=0$ is for $a = e$.

When we start to talk about derivatives, we will see that this is a desirable property. So $f(x) = e^x$ is by far the most common exponential fn.

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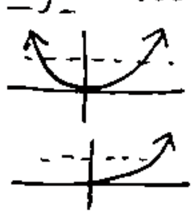
§1.5 Inverse functions and logarithms

Def'n A function $g(x)$ has an inverse function $f = g^{-1}$ if and only if it is one-to-one. In this case, the inverse function $f = g^{-1}$ is defined by $f(y) = x$ if x is the unique element in domain of g such that $g(x) = y$. (f "undoes" g so that $(f \circ g)(x) = x$).

E.g. Since $g(x) = x^3$ is 1-to-1, it admits an inverse $f = g^{-1}$ which is $f(x) = \sqrt[3]{x}$.

E.g. Recall $g(x) = x^2$ is not one-to-one: fails horizontal line test!

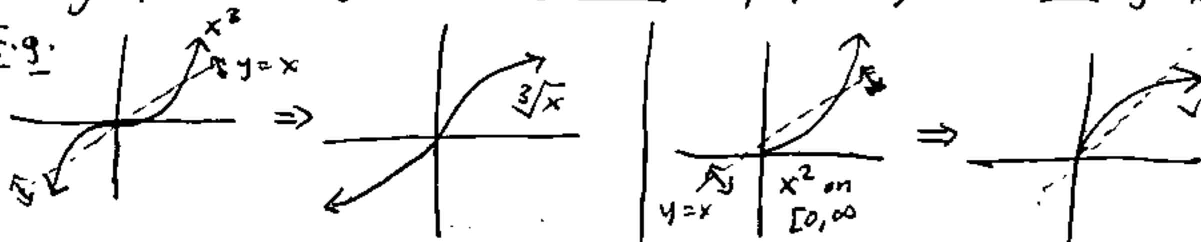


So it does not have an inverse on all of $\mathbb{R}$.

But if we restrict the domain to $[0, \infty)$, then $f(x) = \sqrt{x}$ is its inverse, like we expect.

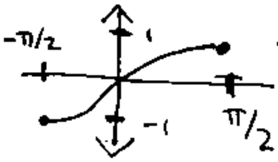
There is a geometric way to think about inverses: the graph of $f = g^{-1}$ is reflection of graph of g over line $y = x$.

E.g.

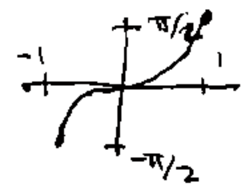


This geometric interpretation also makes clear that domain of f = range of g, and range of f = domain of g for $f = g^{-1}$

E.g. The trig functions are far from 1-to-1, so to define inverse trig functions, we need to restrict domain:



$\sin(x)$ restricted to $[-\pi/2, \pi/2]$ $\Rightarrow$



$\sin^{-1}(x)$ (also called $\arcsin(x)$)

D: $[-1, 1]$

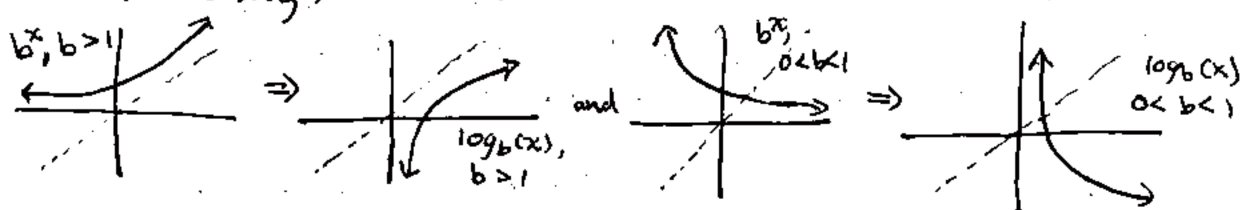
R: $[0, \pi]$

Looking at graph of b^x for any $b>0$, $b \neq 1$, we see it passes horizontal line test, so it has an inverse:

<u>Def'n</u> $\log_b(x)$, the <u>base b logarithm</u>, is the <u>inverse</u> of exponential fn. b^x, meaning $\boxed{\log_b(y) = x \iff b^x = y}$

<u>E.g.</u> $\log_{10}(100) = 2$ since $10^2 = 100$.

Graphically, we have:



Note that since <u>range</u> of b^x is $(0,\infty)$ (positive numbers) <u>domain</u> of $\log_b(x)$ is $(0,\infty)$: can <u>only</u> take <u>log</u> of positive <u>numbers</u>!

Since e^x is the "best" exponential, $\log_e(x)$ is "best" logarithm. It is also called the <u>natural logarithm</u>, denoted $\ln(x) := \log_e(x)$.

Just like we usually only consider e^x for exponential functions, we also usually only consider $\ln(x)$ for logarithms. In fact, these are enough, because of

<u>Thm</u> 1. $b^x = e^{\ln(b)\cdot x}$

2. $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$.

<u>Pf:</u> For 1.: $e^{\ln(b)\cdot x} = (e^{\ln(b)})^x = b^x$. ✓

For 2.: Let $y = \log_b(x)$, so that $b^y = x$.

Taking ln of both sides $\Rightarrow \ln(b^y) = \ln(x)$

$\Rightarrow y \cdot \ln(b) = \ln(x)$

$\Rightarrow \log_b(x) = y = \dfrac{\ln(x)}{\ln(b)}$. ✓ ☒

In the above proof, we used some important properties of exponentials and Logarithms which you hopefully learned in an algebra class:

Prop: 1. $b^{x+y} = b^x b^y$ 2. $b^{x-y} = \frac{b^x}{b^y}$
3. $(b^x)^y = b^{xy}$ 4. $(ab)^x = a^x b^x$

Prop: 1. $\log_b(xy) = \log_b(x) + \log_b(y)$
2. $\log_b(\frac{x}{y}) = \log_b(x) - \log_b(y)$
3. $\log_b(x^r) = r \cdot \log_b(x)$.

Since e^x and $\ln(x)$ are so important, it's also worth remembering these special values:

Prop: 1. $e^0 = 1$ 3. $\ln(1) = 0$
2. $e^1 = e$ 4. $\ln(e) = 1$.

Aside on how to algebraically find inverse function:
To find inverse of $g(x)$, write $y = g(x)$ and "solve for y":

e.g. $g(x) = x^3 - 1 \to y = x^3 - 1$
$y + 1 = x^3$
$\sqrt[3]{y+1} = x$
so inverse $f = g^{-1}$ is $f(y) = \sqrt[3]{y+1}$

e.g. $g(x) = 5e^x \to y = 5e^x$
$\frac{1}{5}y = e^x$
$\ln(\frac{1}{5}y) = x$
so inverse $f = g^{-1}$ is $f(y) = \ln(\frac{1}{5}y)$.

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