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Rules for differentiation §3.1

Now we will spend a lot of time learning rules for derivatives.

The simplest derivative is for a constant function:

Thm If $f(x) = c$ for some constant $c \in \mathbb{R}$,
then $f'(x) = \underline{\underline{0}}$.

Pf: We could write a limit, but it's easier to just remember the tangent line definition of the derivative.
If $y = f(x)$ is a line, then the tangent line at any point is $y = f(x)$. In this case, the slope $= 0$ (horizontal) since $f(x) = c$. ☒

$\longleftrightarrow y = f(x) = c$

Actually, the same argument works for any linear function $f(x)$.

Thm If $f(x) = mx + b$ is a linear function,
then $f'(x) = m$ (slope of line).

Some other simple rules for derivatives are:

Thm
- (sum) $(f+g)'(x) = f'(x) + g'(x)$
- (difference) $(f-g)'(x) = f'(x) - g'(x)$
- (scaling) $(c \cdot f)'(x) = c\, f'(x)$ for $c \in \mathbb{R}$.

Pf: These all follow from the corresponding limit laws.
E.g., for sum rule have

$$(f+g)'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x).$$ ☒

The first really interesting derivative is for $f(x) = x^n$, a power function. We've seen:

$d/dx\,(x^0) = 0$, $d/dx\,(x^1) = 1$, $d/dx\,(x^2) = 2x$

Do you see a pattern?

Thm for any nonnegative integer n, if $f(x) = x^n$ then $\boxed{f'(x) = n \cdot x^{n-1}}$

"bring n down" from the exponent

Pf: We can use an algebra trick:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

check that this multiplies correctly!

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + a x^{n-2} + a^2 x^{n-3} + \dots + a^{n-2} x + a^{n-1})}{x - a}$$

$$= \lim_{x \to a} (x^{n-1} + a x^{n-2} + \dots + a^{n-1}) = \underbrace{a^{n-1} + a^{n-1} + \dots + a^{n-1}}_{n \text{ times}}$$

$$= n \cdot a^{n-1}. \checkmark \quad \blacksquare$$

This is one of the most important formulas in calculus! Please memorize it.

E.g. If $f(x) = 3x^4 - 2x^3 + 6x^2 + 5x - 9$ then

$f'(x) = 12x^3 - 6x^2 + 12x + 5$.

Can easily take derivative of any polynomial!

E.g. If $f(x) = x^3$, what is $f''(x)$?

Well, $f'(x) = 3x^2$, so $f''(x) = 3 \cdot 2x$
$= 6x$.

All derivatives of x^n easy to compute this way! //

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Derivatives for more kinds of functions §3.1

Thm For any real number n, if $f(x) = x^n$
then $\boxed{f'(x) = n \cdot x^{n-1}}$

Exactly same formula as for positive integers n.
Proof is similar, and we will skip it...

E.g. Q: If $f(x) = \sqrt{x}$, what is $f'(x)$?
A: $f(x) = x^{1/2}$, so $f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2}$
$= \frac{1}{2} \frac{1}{x^{1/2}} = \frac{1}{2\sqrt{x}}$.

Q: If $f(x) = \frac{1}{x}$, what is $f'(x)$?
A: $f(x) = x^{-1}$, so $f'(x) = -1 \cdot x^{-1-1} = -x^{-2} = \frac{-1}{x^2}$.

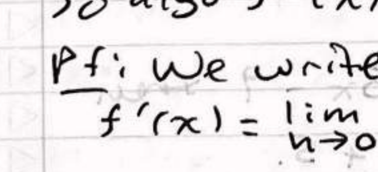
The exponential fn. e^x has a surprisingly simple derivative:

Thm If $f(x) = e^x$, then $f'(x) = e^x = (f(x))$.

Taking derivative of e^x does not change it!
So also $f''(x) = e^x$, $f'''(x) = e^x$, etc...

Pf: We write
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$
$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x \cdot e^0}{h} = e^x \cdot \lim_{h\to 0} \frac{e^h - e^0}{h} = e^x \cdot f'(0)$$

So we just need to show $f'(0) = 1$.

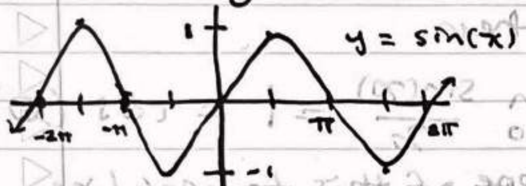


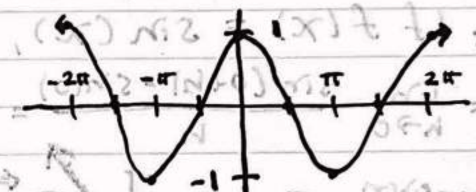
But remember, we defined e as the unique $b>1$ for which slope of tangent of b^x at $x=0$ is one.
So $f'(0) = 1$ by definition of e! ∎

10/2

Derivatives of trigonometric functions §3.3

Looking at the graphs of $\sin(x)$ and $\cos(x)$:





We notice that:

- $\sin(x)$ is increasing $\Leftrightarrow$ $\cos(x) > 0$
- $\sin(x)$ is decreasing $\Leftrightarrow$ $\cos(x) < 0$
- $\sin(x)$ has min./max. $\Leftrightarrow$ $\cos(x) = 0$

- $\cos(x)$ is increasing $\Leftrightarrow$ $\sin(x) < 0$
- $\cos(x)$ is decreasing $\Leftrightarrow$ $\sin(x) > 0$
- $\cos(x)$ has min./max. $\Leftrightarrow$ $\sin(x) = 0$

From these qualitative properties, reasonable to guess:

<u>Thm</u> $\boxed{d/dx(\sin(x)) = \cos(x)}$
and $\boxed{d/dx(\cos(x)) = -\sin(x)}$

E.g.: If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$,
so $f''(x) = -\sin(x)$, and $f'''(x) = -\cos(x)$,
and $f^{(4)}(x) = -(-\sin(x)) = \sin(x) = f(x)$.
After 4 derivatives, we get back what we started with!
Can also check that if $f(x) = \cos(x)$, then $f^{(4)}(x) = \cos(x) = f(x)$.

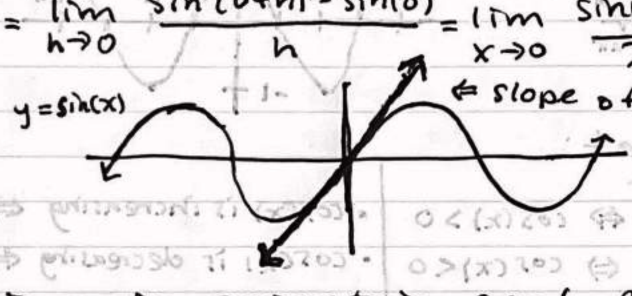
In this way, the trig functions $\sin(x)$ and $\cos(x)$ behave like e^x, where taking enough derivatives gives us back the original function we started with.

Whereas with a polynomial function like
$f(x) = 5x^4 - 3x^3 + 6x^2 + 10x - 9$, taking enough derivatives always gives us zero!

The key step for proving $d/dx\,(\sin(x)) = \cos(x)$ is this:

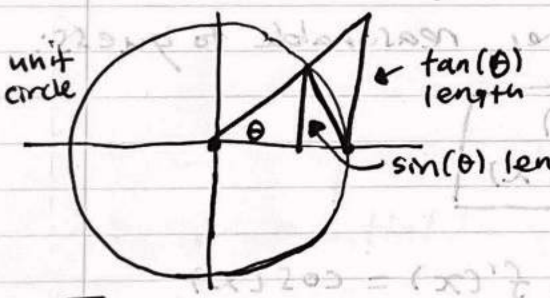
Lemma If $f(x) = \sin(x)$, then

$$f'(0) = \lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{x \to 0} \frac{\sin(x)}{x} = 1 = \cos(0).$$



⇐ slope of this tangent line at $x = 0$ should be $= \underline{\underline{1}}$.

There is a nice geometric proof of this Lemma:



⇐ idea of proof is to compare areas of triangles in this drawing. See book for details!

For our purposes, we will just use the formulas. To summarize, it is worth memorizing the following important derivatives:

$$\boxed{\begin{array}{ll} d/dx\,(x^n) = n \cdot x^{n-1} & d/dx\,(\sin(x)) = \cos(x) \\ d/dx\,(e^x) = e^x & d/dx\,(\cos(x)) = -\sin(x) \end{array}}$$

don't forget this <u>negative</u> sign: it's important!

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The product and quotient rules §3.2

Suppose we want to take the derivative of a product $f \cdot g$ of two (differentiable) functions $f(x)$ and $g(x)$.
Might think/hope its derivative is product of derivatives,
but easy to find examples where $(f \cdot g)'(x) \neq f'(x) \cdot g'(x)$.

E.g. Let $f(x) = x$, $g(x) = x^2$, then $f'(x) \cdot g'(x) = 1 \cdot 2x = 2x$,
but $(f \cdot g)(x) = x^3$, so $(fg)'(x) = 3x^2$.

Instead, we have the product rule:

Thm For two (differentiable) functions $f(x), g(x)$:

$$\boxed{(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)}$$

"First times derivative of second plus second times derivative of first."

In differential notation: $\boxed{d/dx\,(f \cdot g) = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}}$

E.g. With $f(x) = x$ and $g(x) = x^2$, we compute

$$(f \cdot g)'(x) = f(x)\,g'(x) + g(x)\,f'(x) = x \cdot 2x + x^2 \cdot 1$$
$$= 3x^2 = d/dx\,(x^3) \checkmark$$

E.g. $d/dx\,(xe^x) = x\, d/dx\,(e^x) + e^x\, d/dx\,(x) = xe^x + e^x$

E.g. $d/dx\,(x^2 \sin(x)) = x^2\, d/dx\,(\sin(x)) + \sin(x)\, d/dx\,(x^2)$
$= x^2 \cos(x) + 2x \sin(x)$

Pf sketch for product rule:
Write $u = f(x)$, $v = g(x)$, $\Delta u = f(x+h) - f(x)$, $\Delta v = g(x+h) - g(x)$.
Then $\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv$
$= u\Delta v + v\Delta u - \underbrace{\Delta u\, \Delta v}_{\text{this term goes away in limit!}}$

See book for details!

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The quotient rule is a bit more complicated:

Thm For two (differentiable) functions $f(x), g(x)$ (with $g(x) \neq 0$),

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot f'(x) - f(x)\, g'(x)}{g(x)^2}},$$ or in differential notation:

$$d/dx\left(\frac{f}{g}\right) = \left(g\frac{df}{dx} - f\frac{dg}{dx}\right) / g^2.$$

—

Looks similar in many ways to product rule, but more complicated. When we learn the chain rule, you will see that you don't need to separately memorize the quotient rule.

—

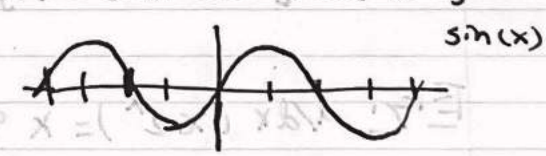
E.g. Let $f(x) = x$, $g(x) = 1-x$, so $\frac{f}{g}(x) = \frac{x}{1-x}$

Then $\boxed{\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot f'(x) - f(x)\, g'(x)}{g(x)^2} = \frac{(1-x)\cdot 1 - x\cdot(-1)}{(1-x)^2}}$

$$= \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

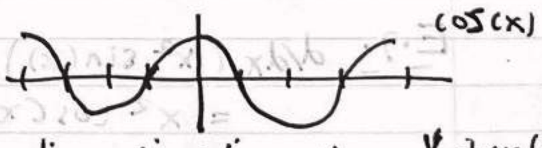
Any rational function can be differentiated this way...

E.g. Recall $\tan(x) = \frac{\sin(x)}{\cos(x)}$

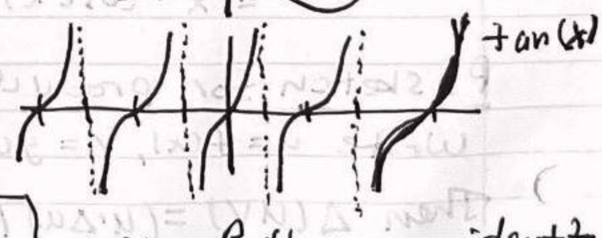


Thus, $\boxed{(\tan)'(x)} =$

$$\frac{\cos(x)\cdot(\sin)'(x) - \sin(x)\cos'(x)}{(\cos(x))^2}$$



$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$



$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \boxed{\frac{1}{\cos^2(x)}}$$

using Pythagorean identity: $\boxed{\sin^2(x) + \cos^2(x) = 1}$ for any x

the one trig identity really worth knowing

10/6

Chain rule §3.4

Let $f(x) = \sqrt{x^2+1}$. How do we find $f'(x)$?
So far we don't know how... to do this we need the chain rule, which tells us how to take derivatives of compositions of functions:

Theorem For two (differentiable) fn's $f(x)$ and $g(x)$, we have $\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$

In differential notation, this can be written
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
where $y = f(g(x))$ and $u = g(x)$. So roughly speaking the chain rule lets us "cancel" the du's.

E.g. For $f(x) = \sqrt{x^2+1}$, write $f(x) = h(g(x))$ where $h(x) = \sqrt{x}$ and $g(x) = x^2+1$. Then the chain rule says $f'(x) = h'(g(x)) \cdot g'(x)$
$$= \tfrac{1}{2}(g(x)^{-\frac{1}{2}}) \cdot 2x = \frac{x}{\sqrt{x^2+1}}.$$

E.g. Let $f(x) = \sin(x^2)$. Then
$$f'(x) = \underbrace{\cos(x^2)}_{d/dx(\sin(x)),\ \text{plug in } x^2} \cdot \underbrace{2x}_{d/dx(x^2)}$$

E.g. Let $f(x) = \sin^2(x)$ (meaning $(\sin(x))^2$).

Then $f'(x) = \underbrace{2 \cdot \sin(x)}_{d/dx(x^2),\ \text{plug in } \sin(x)} \cdot \underbrace{\cos(x)}_{d/dx(\sin(x))}$.

E.g. with $f(x) = \frac{1}{\sin(x)} = (\sin(x))^{-1}$, we have

$$f'(x) = \underbrace{-(\sin(x))^{-2}}_{d/dx(x^{-1}),\ \text{plug in } \sin(x)} \cdot \underbrace{\cos(x)}_{d/dx(\sin(x))} = \frac{-\cos(x)}{\sin^2(x)}$$

With this last example, we see how we don't need the quotient rule. In fact, quotient rule can be deduced from product rule and chain rule:

Let $h(x) = \frac{f(x)}{g(x)} = f(x) \cdot (g(x))^{-1}$

Then $h'(x) = f(x) \cdot d/dx(g(x)^{-1}) + \frac{1}{g(x)} f'(x)$.

But by the chain rule,

$$d/dx(g(x)^{-1}) = -g(x)^{-2} \cdot g'(x) = \frac{-g'(x)}{g(x)^2}$$

So that $h'(x) = f(x) \cdot \frac{-g'(x)}{g(x)^2} + \frac{f'(x)}{g(x)}$

$$= \frac{-f(x) \cdot g'(x)}{g(x)^2} + \frac{f'(x)\, g(x)}{g(x)^2}$$

$$= \frac{g(x) f'(x) - f(x) g'(x)}{g(x)^2},$$

which is exactly the quotient rule we learned. So you never need to separately memorize the quotient rule: the product rule and chain rule are enough. //

10/10

§3.4, 3.6

<u>Derivatives</u> <u>of</u> <u>exponentials</u> <u>and</u> <u>logarithms</u>

The chain rule allows us to compute derivatives of arbitrary exponential and logarithmic functions.

Let's start with the exponential $f(x) = b^x$ for some base $b > 0$. Recall that

$$b^x = e^{\ln(b) \cdot x} \quad \text{by rules of exponents.}$$

Thus $d/dx\,(b^x) = d/dx\,(e^{\ln(b) \cdot x})$

$$= e^{\ln(b) \cdot x} \cdot \ln(b) \quad \text{by chain rule}$$
$$= \ln(b) \cdot b^x$$

So derivative behaves similarly to e^x (but w/ $\ln(b)$ factor)...

What about logarithms? Recall that

$$x = e^{\ln(x)} \quad \text{(because } e^x \text{ and } \ln(x) \text{ are inverses...)}$$

Taking d/dx of both sides gives:

$$d/dx\,(x) = d/dx\,(e^{\ln(x)})$$
$$1 = e^{\ln(x)} \cdot d/dx\,(\ln(x)) \quad \text{by chain rule}$$
$$1 = x \cdot d/dx\,(\ln(x))$$
$$\Rightarrow \boxed{d/dx\,(\ln(x)) = \frac{1}{x}}$$

How about arbitrary logarithms?

If $f(x) = \log_b(x)$ for some base $b > 0$,

then since $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ by rules of logs

we have $f'(x) = \dfrac{1}{\ln(b)} \cdot \dfrac{1}{x}$.

Notice: You might expect that there is some power function $f(x) = a \cdot x^n$ with $f'(x) = 1/x = x^{-1}$. But we would need $n = 0$ and $a = "\frac{1}{0}"$ for this to work ($f'(x) = a \cdot n \cdot x^{n-1}$), so there is not such an $f(x)$!

Now that we know:

- Sum, difference, scaling rules:
  $$d/dx\,(c \cdot f(x) + d \cdot g(x)) = c \cdot f'(x) + d \cdot g'(x)$$
- product rule:
  $$d/dx\,(f(x) \cdot g(x)) = f(x)\,g'(x) + g(x)\,f'(x)$$
  (and maybe quotient rule...)
- chain rule:
  $$d/dx\,(f(g(x))) = f'(g(x)) \cdot g'(x).$$
- and derivatives of basic functions:
  $d/dx\,(x^n) = n \cdot x^{n-1}$ for any $n \in \mathbb{R}$
  $d/dx\,(e^x) = e^x$ and $d/dx\,(\ln(x)) = 1/x$
  $d/dx\,(\sin(x)) = \cos(x)$ and $d/dx\,(\cos(x)) = -\sin(x)$,

We can compute the derivative of essentially any kind of function that we have been studying all semester!

Exercise: Find $d/dx\,(\sin(\ln(x^2)))$.

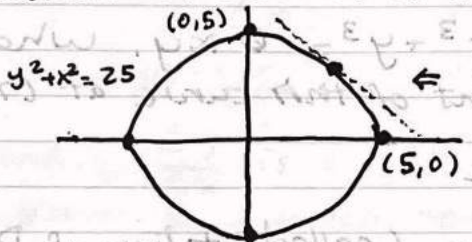
10/13

Implicit differentiation §3.5

We've been studying curves of the form $y = f(x)$.
But we can also consider equations like

$$(*) \quad y^2 + x^2 = 25$$

where y is defined "implicitly" in terms of x.
The equation $(*)$ defines a circle of radius 5:



Even though this is not exactly the graph of a function (it doesn't pass the vertical line test), we can still make sense of the derivative $y' = dy/dx$ at any point (x, y) on this curve: we can still consider the slope of the tangent to the curve at (x,y).

How can we find $\frac{dy}{dx}$ when y is defined implicitly in terms of x? It turns out we can use the chain rule to do this without having to solve for y in terms of x!

E.g. What is the slope of the tangent to the circle $x^2+y^2 = 25$ at the point $(x,y) = (3,4)$?

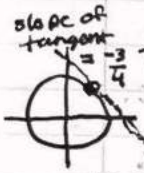
Let's use implicit differentiation to answer this. This means we take the equation

$$x^2 + y^2 = 25$$

and apply d/dx to both sides of it:

$$d/dx\,(x^2 + y^2) = d/dx\,(25)$$
$$d/dx\,(x^2) + d/dx\,(y^2) = 0$$
$$2x + \underbrace{2y \cdot dy/dx}_{} = 0$$

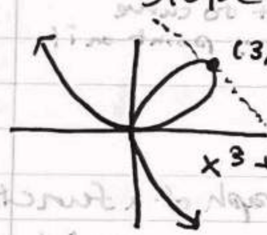
← this part we got from the chain rule!



Then we solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

At $(x, y) = (3, 4)$ this gives $dy/dx = -3/4$. ✓

E.g. Find y' if $x^3 + y^3 = 6xy$. What is slope to tangent of this curve at $(x, y) = (3, 3)$?



$x^3 + y^3 = 6xy$ (called 'folium of Descartes')

To do this, we implicitly differentiate $x^3 + y^3 = 6xy$:
$$d/dx\,(x^3 + y^3) = d/dx\,(6xy)$$
$$3x^2 + d/dx\,(y^3) = \underbrace{6x \cdot d/dx(y) + y \cdot d/dx(6x)}_{\text{product rule}}$$
$$3x^2 + \underbrace{3y^2 \frac{dy}{dx}}_{\text{chain rule}} = 6x\frac{dy}{dx} + 6y$$

Solve for dy/dx: $\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$

$$\text{So } y' = \frac{dy}{dx} = \frac{6y - 3x^2}{3y - 6x} = \frac{2y - x^2}{y^2 - 2x}$$

At $(x, y) = (3, 3)$ this gives:
$$\frac{dy}{dx} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$$ (looks correct on graph)

Note: No way we could solve $x^3 + y^3 = 6xy$ for y, (unlike circle example) so we have to differentiate implicitly.

10/16

Rates of change in the sciences §3.7

Let's take a minute to review the importance of the derivative to the sciences more broadly.

Suppose $y = f(x)$ models some phenomenon:
recall x is independent variable and y dependent variable.
(We think of y as being "determined" by x.)

The change in x $\Delta x = x_2 - x_1$ from x_2 to x_1
causes a change in y $\Delta y = y_2 - y_1$ where $y_2 = f(x_2)$ & $y_1 = f(x_1)$.

The quantity $\frac{\Delta y}{\Delta x}$ is the (average) rate of change:
it represents how much "output" changes in response to a change in the "input."

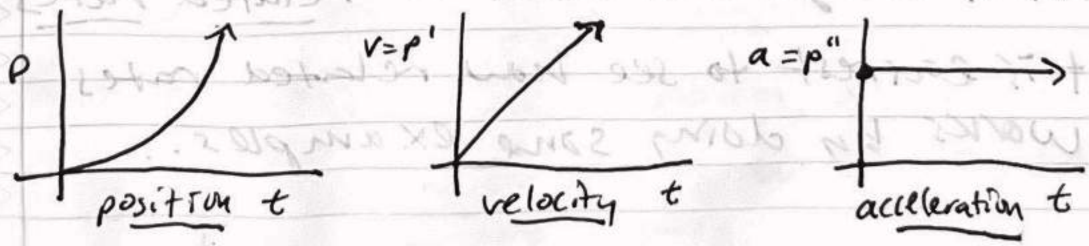
The quantity $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ (the derivative)
is the instantaneous rate of change.

E.g. Physics: velocity and acceleration

We've already explained several times that if
$p = f(t)$ is the position of something (e.g. car or particle)
as a function of time t, then:

$$V = p' = \frac{dp}{dt} \text{ is the velocity (speed) at time } t$$

and $$a = p'' = \frac{d^2 p}{dt^2} \text{ is the acceleration at time } t.$$



E.g. Economics: marginal cost (or revenue, etc.)
If $y = f(x)$ represents the total cost for a firm to produce x units of a product, the derivative dy/dx = marginal cost, cost of producing one new unit.
(Notice that the independent variable here is not time!)

E.g. Biology: population growth
If $n = f(t)$ is the size (# of organisms) of a population at time t, then $\frac{dn}{dt}$ = (instantaneous) growth rate, telling us rate population is growing or shrinking.

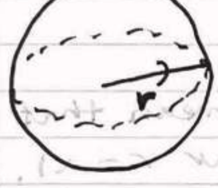
Related rates §3.9

Suppose that we have two functions $f(t)$ and $g(t)$ (where the independent variable t represents time, say). It may be easier to measure how one of them, $g(t)$, is changing over time, but we may really care about how the other one, $f(t)$, is changing.
If the two functions $f(t)$ and $g(t)$ are related in some way (say, by geometry...) then their rates of change are also related (by using the chain rule)
This is the general idea of related rates.
It is easiest to see how related rates works by doing some examples...

E.g. Suppose that a spherical balloon is filling with air.

Let $V(t)$ = volume of balloon (in cm^3) at time t (in s)

and $r(t)$ = radius of balloon (in cm) at time t



It is probably easier to measure the volume, but perhaps we really want to know how the radius is changing over time.

Suppose that $\boxed{\frac{dV}{dt} = 100 \ cm^3/s}$ ← Given

i.e., volume is increasing at constant rate of $100 \ cm^3/s$

What is the rate at which the radius is increasing when the radius is $r = 25$ cm?

i.e., $\boxed{\text{What is } \frac{dr}{dt} \text{ when } r = 25 \text{ cm?}}$ ← Want to find

To answer, we need to know how volume is related to radius.

So recall that the volume of a sphere is given by:

$$V = \frac{4}{3}\pi r^3$$

Then, to figure out how $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are related, differentiate:

$$d/dt\,(V) = d/dt\,(4/3\ \pi\ r^3)$$

$$dV/dt = 4/3\ \pi\ 3r^2\ dr/dt \quad \leftarrow \text{remember chain rule!}$$

$$\Rightarrow \ dr/dt = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2}.$$

With $\frac{dV}{dt} = 100 \ cm^3/s$ and $r = 25$ cm, we get

$$\frac{dr}{dt} = 100 \cdot \frac{1}{4\pi(25)^2} = \frac{1}{25\pi} \approx 0.0127 \ cm/s.$$

cm/s cm/s

//

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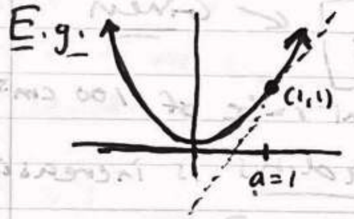
Linear approximation § 3.10

Let $f(x)$ be a function differentiable at $x=a$.
The tangent line to the curve $y=f(x)$ at $(x,y)=(a,f(a))$ is the best linear approximation to $f(x)$ near $x=a$.
Its equation is given by

$$\boxed{L(x) = f(a) + (x-a)\cdot f'(a)}$$

We write "$f(x) \approx f(a) + (x-a)\cdot f'(a)$" to mean that $f(x)$ approximately follows this line (near $x=a$).

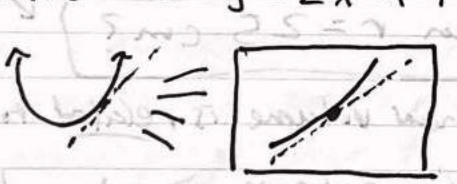
E.g.



$y=f(x)=x^2$
equation of tangent line to $y=x^2$ at the point $(x,y)=(1,1)$ is:

$$L(x) = f(a) + (x-a)\cdot f'(a)$$
$$= 1 + (x-1)\cdot 2 = 2x-1$$

So the line $y=2x-1$ is "close" to $y=x^2$ at x values near $x=1$.



In general, if we "zoom in" to the curve $y=f(x)$ at $(x,y)=(a,f(a))$ the curve will look very close to the tangent line $y=f(a)+(x-a)\cdot f'(a)$.

The linear approximation given by the tangent line is useful because in many applied situations we may be able to compute $f(a)$ and $f'(a)$ at point $x=a$, but $f(x)$ may be very complicated.
So $L(x) = f(a) + (x-a)\cdot f'(a) \approx f(x)$ is easier to work with.

Sometimes use "differentials" to represent linear approximation:

$$\boxed{dy = f'(x)\cdot dx} \quad (\text{think: } \tfrac{dy}{dx} = f'(x))$$

The linear approximation is then:

$$\boxed{\underbrace{\Delta y}_{(f(x)-f(a))} \approx f'(x)\cdot \underbrace{\Delta x}_{(x-a)}}$$