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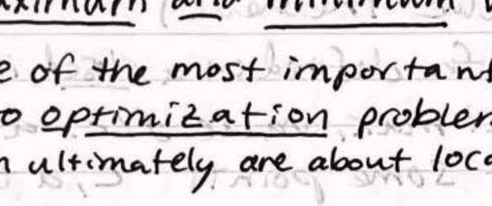
Maximum and minimum values §4.1

One of the most important applications of calculus is to optimization problems: finding "best" option, which ultimately are about locating maxima and minima.

Def'n Let c be in domain of function f. We say $f(c)$ is:

- absolute (or global) maximum if $f(c) \geq f(x)$ $\forall$ x in domain of f,
- absolute (or global) minimum if $f(c) \leq f(x)$ $\forall$ x in domain,
- local maximum if $f(c) \geq f(x)$ for x "near" c,
- local minimum if $f(c) \leq f(x)$ for x "near" c.

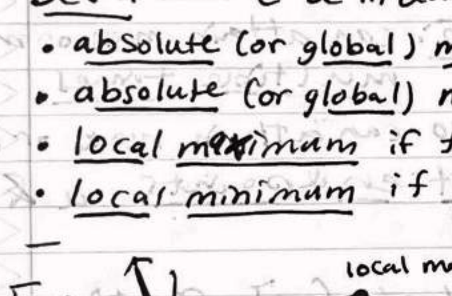
E.g.

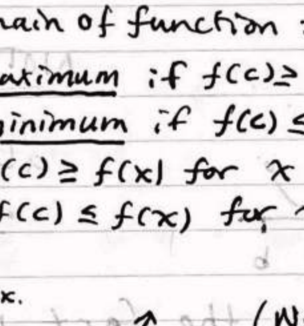


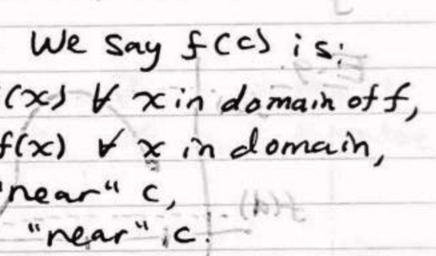
(Note that global min./max. are also always local min./max.)

The behavior of min./max. for functions $f: \mathbb{R} \to \mathbb{R}$ can be very complicated, even for the "nice" functions we've been looking at:

E.g.







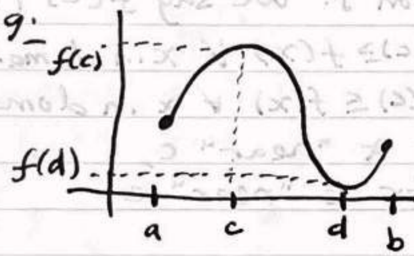
And of course we saw above how local min. & max. do not need to be global min. & max.

Things are much better when we restrict the domain of f to be a closed interval $[a, b]$:

global min./max. are also called "extreme values"

Theorem (Extreme Value Theorem)
Let f be a continuous fn. defined on a closed interval $[a,b]$. Then f attains a global max. value $f(c)$ and a global min. value $f(d)$ at some points $c, d \in [a,b]$.

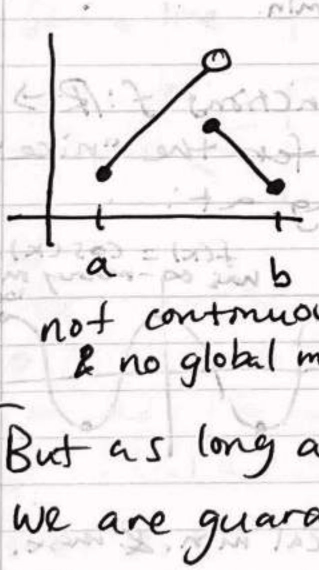
E.g.:



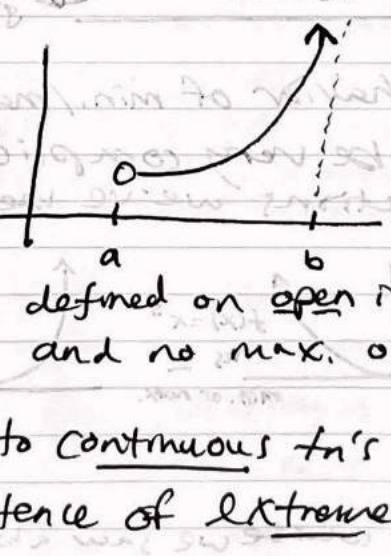
NOTE: can attain max. or min. multiple times

also can attain max. or min. at endpoints a & b.

WARNING: Both the fact that f is continuous & fact that its domain is a closed interval are crucial for the Extreme Value Thm.



not continuous & no global max.



defined on open interval (a,b) and no max. or min.

But as long as we stick to continuous fn's on closed intervals, we are guaranteed existence of extreme values.

But... how do we find the location of the extreme values that we know must exist?

We use calculus! Specifically: the derivative!

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We mentioned before that at (local) min./max., the derivative must be zero.

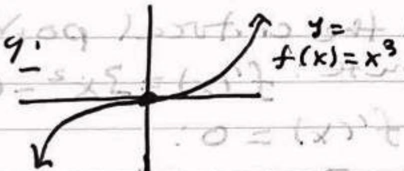
Thm (Fermat) If f has local min./max. at c, and if $f'(c)$ exists, then $f'(c) = 0$.

E.g.

← intuitive from tangent line slope definition of derivative.

WARNING: The converse of this theorem is not true, i.e., if $f'(c) = 0$ it does not mean c is location of min./max.

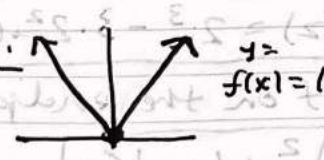
E.g.



For $f(x) = x^3$ we have $f'(0) = 0$ (since $f'(x) = 3x^2$), but 0 is not a local min./max. since there are no local min./max..

WARNING: If $f'(c)$ does not exist, c could be location of a local min./max.!

E.g.



For $f(x) = |x|$ (absolute value) we explained before why $f'(0)$ does not exist, but 0 is a global minimum!

DEF'N A critical point (or critical number) of a function $f(x)$ is a point $x = c$ where either:

- $f'(c) = 0$
- or $f'(c)$ does not exist.

We can use critical points to find extreme values:

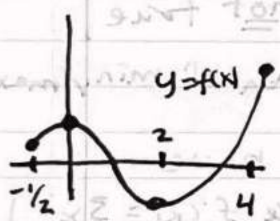
§4.1

The Closed Interval Method

To find the absolute minimum and maximum of a continuous function f defined on a closed interval $[a,b]$:

1. Find the values of f at the critical points of f in (a,b).
2. Find the values of f at the endpoints of interval (i.e., $f(a)$ and $f(b)$).
3. The largest value from steps 1 & 2 is the abs. max. The smallest value from steps 1 & 2 is the abs. min.

E.g. Problem: Find the absolute maximum and minimum of $f(x) = x^3 - 3x^2 + 1$ on interval $-1/2 \le x \le 4$



Solution: We use the Closed Interval Method.

1. We need to find the critical points. So we compute: $f'(x) = 3x^2 - 6x$ and solve for $f'(x) = 0$:

$$3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 2.$$

The critical points are $x=0$ and $x=2$. Their f values are:

$\boxed{f(0) = 0^3 - 3\cdot 0^2 + 1 = 1}$ and $\boxed{f(2) = 2^3 - 3\cdot 2^2 + 1 = -3}$

2. We compute the values of f on the endpoints:

$$\boxed{f(-1/2) = (-1/2)^3 - 3\cdot(-1/2)^2 + 1 = 1/8}$$

and $\boxed{f(4) = 4^3 - 3\cdot 4^2 + 1 = 17}$

3. The abs. max. is the largest circled # above: i.e., $\boxed{\max = 17}$ which occurs $\boxed{\text{at } x = 4}$

The abs. min. is the smallest circled # above: i.e., $\boxed{\min = -3}$ which occurs $\boxed{\text{at } x = 2}$

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The Mean Value Theorem and its consequences §4.2

The IVT and EVT are important results about continuous f.

The Mean Value Theorem is a 3rd important result for differentiable f.

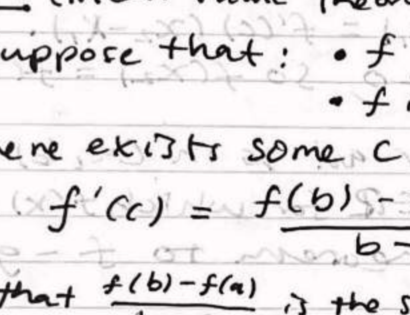
Theorem (Mean Value Theorem) Let f be defined on $[a,b]$ and suppose that:
- f is continuous on $[a,b]$
- f is differentiable on (a,b)

Then there exists some c is (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Notice that $\frac{f(b)-f(a)}{b-a}$ is the slope of line from $(a,f(a))$ to $(b,f(b))$.

Picture:

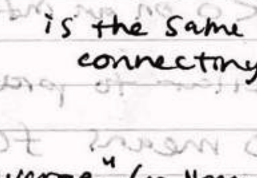


$\Leftarrow$ the mean value theorem says there is some point c where the slope of the tangent is the same as the slope of line connecting the endpoints

Since $\frac{f(b)-f(a)}{b-a}$ is also the "average" (or "mean") rate of change of f, MVT can also be thought of as saying somewhere on interval instantaneous rate of change = average rate of change.

Pf idea: Case where $f(a)=f(b)$ is called Rolle's Theorem.

It says that if f looks like: then it has a local min. or max. in (a,b), which follows from EVT.



More general case when $f(a) \neq f(b)$ follows by "tilting your head" ☒

The Mean Value Theorem has many important consequences ...

Thm If $f'(x) = 0$ for all x in (a,b), then f is constant on all of (a,b).

Pf: Choose any points $x_1 < x_2$ in (a,b). Then by the MVT, there is some c with $x_1 < c < x_2$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. But by assumption $f'(c) = 0$, so $f(x_2) = f(x_1)$. $\boxtimes$

Cor If $f'(x) = g'(x)$ for all x in (a,b), then there is a constant $C \in \mathbb{R}$ for which $f(x) = g(x) + C$.

Pf: Apply previous theorem to $f - g$. $\boxtimes$

What the derivative says about shape of graph §4.3

Thm • If $f'(c) > 0$ on an interval, then f is increasing (on this interval)

• If $f'(c) < 0$ on an interval, then f is decreasing.

Pf: Very similar to proof of last theorem, but now $f'(c) > 0$ means $f(x_2) > f(x_1)$ (increasing) $\boxtimes$

This can help us draw graph of f:

E.g. Consider $f(x) = x^3 - 3x$, so $f'(x) = 3(x^2 - 1) = 3(x+1)(x-1)$.

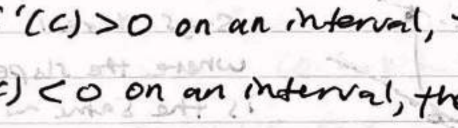
We know the critical points are $x = -1$ and $x = 1$.

Choose points "inbetween" the C.P.'s:

$x = -2 \Rightarrow f'(-2) = 3(4-1) = 9 > 0$

$x = 0 \Rightarrow f'(0) = 3(-1) = -3 < 0$

$x = 2 \Rightarrow f'(2) = 3(4-1) = 9 > 0$



← "sign chart" for $f'(x)$

$\Rightarrow$ So f is increasing on $(-\infty, -1)$, decreasing on $(-1, 1)$, increasing on $(1, \infty)$. $\boxtimes$

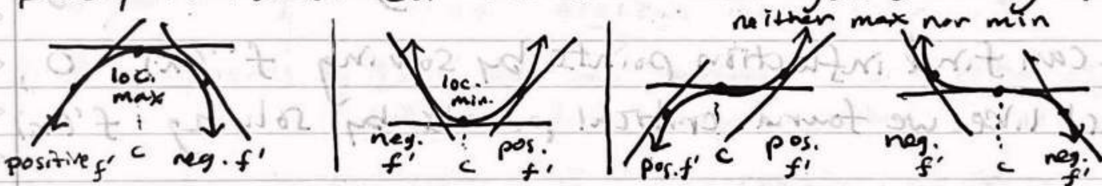
10/27 §4.3

The sign of $f'(x)$ dictating increasing vs. decreasing also means we can use derivative to identify local min. & max.

Thm (First Derivative Test) Let c be a critical point of f.

1) If f' changes from negative to positive at c, c is a local min.
2) If f' changes from positive to negative at c, c is a local max.
3) If f' has same sign to left and right of c (i.e., both positive or both negative) then c is not a local min. or max.

Easy to remember this criterion if you draw graph:



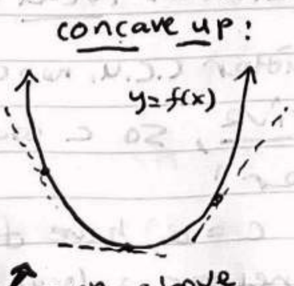
E.g. With $f(x) = x^3 - 3x$ as before, we found the sign chart of $f'(x)$ to be: $\overset{+}{}\ \underset{-1}{\overset{0}{|}}\ \overset{-}{}\ \underset{1}{\overset{0}{|}}\ \overset{+}{}$

So -1 is a local min, and 1 a local max.

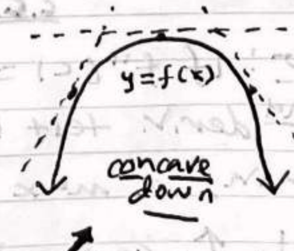
The second derivative f'' also tells us about shape of graph:

DEF'N If on some interval, the graph of f lies above all its tangents, then we say f is concave up on this interval. If on an interval, the graph of f lies below all its tangents, then f is concave down on this interval.

E.g. concave up:



graph above tangent lines



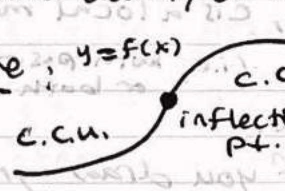
graph below tangent lines

Thm. • If $f''(x) > 0$ on an interval, then f is <u>concave up</u> there.

• If $f''(x) < 0$ on an interval, then f is <u>concave down</u> there

<u>DEF'N</u> A point where f switches from concave up to concave down, or vice-versa, is called an <u>inflection point</u>.

<u>Picture</u>:



$\Leftarrow$ at the inflection point, <u>rate</u> <u>of</u> <u>growth</u> switches from increasing to decreasing!

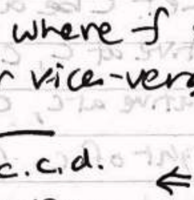
We can find inflection points by solving $f''(x) = 0$, just like we found critical points by solving $f'(x) = 0$.

The second derivative also can identify min.'s/max's!

<u>Thm</u> (Second Derivative Test) Let c be a critical point of f.

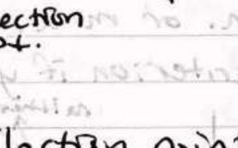
- If f is <u>concave</u> <u>up</u> at c, then c is a <u>local</u> <u>min</u>.
- If f is <u>concave</u> <u>down</u> at c, then c is a <u>local</u> <u>max</u>.

<u>E.g.</u> $f(x) = x^2 \Rightarrow$



$c = 0$ is a c.p. and $f''(0) = 2 > 0$ so c.c.u. $\Rightarrow$ local min

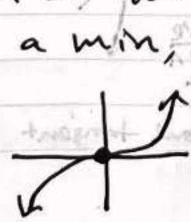
$f(x) = -x^2 \Rightarrow$



$c = 0$ is a c.p. and $f''(0) = -2 < 0$ so c.c.d $\Rightarrow$ local max.

<u>WARNING</u>: If $f''(c) = 0$ (so f is neither c.c.u. nor c.c.d. at c) then 2^{nd} deriv. test is <u>inconclusive</u>, so c could be a min, a max, or neither!

<u>E.g.</u>



$f(x) = x^3 \Rightarrow$ at c.p. $c = 0$, have $f''(c) = 0$ and 0 is <u>neither</u> a local min. nor local max!

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Summary of curve sketching §4.5

Now that we have the tools of the 1^{st} and 2^{nd} derivatives, we can produce reasonable sketches of graphs of most f.

Let's summarize the main things to depict in sketches of f(x):

A Domain - where is f(x) defined?

B Intercepts - Where does graph cross x- and y-axes? i.e., where is $f(x)=0$ and what is $f(0)$?

C Symmetry and periodicity - Is f(x) even or odd? Is it periodic (like sin/cos)?

D Asymptotes - Does f(x) have horizontal or vertical asymptotes? Where? Recall these are limits at, or =, ∞.

E Increasing/Decreasing - Where is f(x) increasing or decreasing? To answer this we look at $f'(x)$, where it is >0 or <0.

F (Local) Minima/Maxima - Where are the min./max. of f(x)? What are their values? Use critical points ($f'(x)=0$) to find these.

G Concavity & points of inflection - where is the graph of f(x) concave up or concave down? Where are inflection points? Use $f''(x)$ (where it is >0, <0, or $=0$) to find these.

Eg. Let's use these guidelines to sketch graph of

$$f(x) = e^{-\frac{x^2}{2}}.$$

A Domain: f(x) is defined on all of $\mathbb{R}$.

B Intercepts: $f(0) = e^0 = 1$, and this is only intercept, because $e^{\text{anything}} > 0$.

C Symmetry – Since x^2 is even, $f(x)$ is even, i.e. $f(-x) = f(x)$, i.e. symmetric over y-axis.

D Asymptotes – Since $\lim_{x \to -\infty} e^x = 0$, we have that

$\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0$, i.e., horizontal asymptote at $y = 0$

E Increasing/ Decreasing – We compute $f'(x) = d/dx\,(e^{-\frac{x^2}{2}})$ chain rule

$= e^{-x^2/2} \cdot d/dx\,(\frac{-x^2}{2})$

$= -x \cdot e^{-x^2/2}$

Since $e^{\text{anything}} > 0$, we have that

$\underbrace{f'(x) > 0}_{f(x)\text{ increasing}}$ for $x < 0$ and $\underbrace{f'(x) < 0}_{f(x)\text{ decreasing}}$ for $x < 0$

F min./max. – Solving $f'(x) = 0 \Rightarrow -x \cdot e^{-x^2/2} = 0$

$\Rightarrow x = 0$ (since $e^{\text{anything}} > 0$)

So 0 is the only c.p., and it is a local max by 1st derivative test ($f'(x)$ goes from + to − at 0)

G Concavity and inflection – Compute $f''(x) = d/dx\,(-x \cdot e^{-x^2/2})$ product rule

$= -x \cdot d/dx\,(e^{-x^2/2}) + e^{-x^2/2} \cdot d/dx\,(-x)$

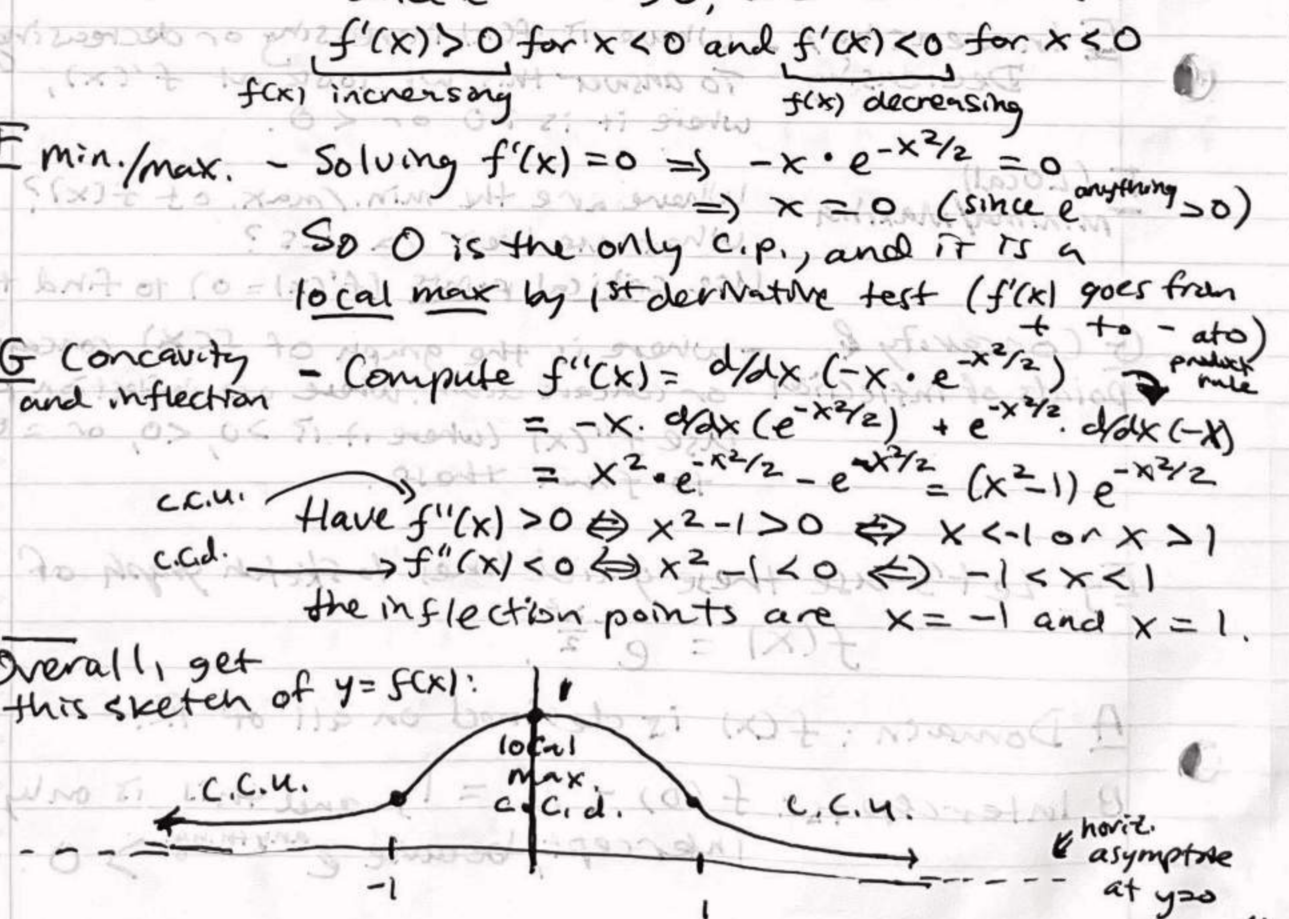
$= x^2 \cdot e^{-x^2/2} - e^{-x^2/2} = (x^2 - 1)\,e^{-x^2/2}$

c.c.u. Have $f''(x) > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow x < -1$ or $x > 1$

c.c.d. $f''(x) < 0 \Leftrightarrow x^2 - 1 < 0 \Leftrightarrow -1 < x < 1$

the inflection points are $x = -1$ and $x = 1$.

Overall, get this sketch of $y = f(x)$:



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L'Hôpital's Rule §4.4

Recall the derivative was defined as a limit.
The derivative can also help us compute certain limits.
The kinds of limits the derivative helps with are the "indeterminate forms," meaning "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$".

Def'n A limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is said to be of indeterminate form of type $\frac{0}{0}$ if $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$.

E.g. $\lim_{x \to 1} \frac{\ln(x)}{x-1}$ is indeterminate of type $\frac{0}{0}$ since $\ln(1) = 0$ and $1 - 1 = 0$.

This is a limit we cannot evaluate just by "plugging in."

Def'n A limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is indeterminate of type $\frac{\infty}{\infty}$ if $\lim_{x \to a} f(x) = \pm\infty$ and also $\lim_{x \to a} g(x) = \pm\infty$.

E.g. $\lim_{x \to \infty} \frac{\ln(x)}{x-1}$ is indeterminate of type $\frac{\infty}{\infty}$ since $\lim_{x \to \infty} \ln(x) = \infty$ and $\lim_{x \to \infty} x - 1 = \infty$.

Theorem (L'Hôpital's Rule) If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ then $\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$.

Note: Here we also allow $a = \pm\infty$ (limits at ∞) or one-sided limits like $\lim_{x \to a^+} \frac{f(x)}{g(x)}$, etc.

E.g. Since $\lim_{x\to 1} \frac{\ln(x)}{x-1}$ is indeterminate of type $\frac{0}{0}$, we can apply L'Hôpital's Rule to compute:

$$\lim_{x\to 1} \frac{\ln(x)}{x-1} = \lim_{x\to 1} \frac{d/dx(\ln(x))}{d/dx(x-1)} = \lim_{x\to 1} \frac{1/x}{1} = \lim_{x\to 1} \frac{1}{x} = \underline{\underline{1}}.$$

E.g. Since $\lim_{x\to\infty} \frac{\ln(x)}{x-1}$ is indeterminate of type $\frac{\infty}{\infty}$, we can apply L'Hôpital:

$$\lim_{x\to\infty} \frac{\ln(x)}{x-1} = \lim_{x\to\infty} \frac{d/dx(\ln(x))}{d/dx(x-1)} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = \underline{\underline{0}}$$

<u>WARNING</u>: L'Hôpital's Rule <u>does</u> <u>not</u> <u>work</u> if the limit is <u>not</u> of indeterminate form.

E.g. If we tried to apply L'Hôpital to $\lim_{x\to 0} \frac{x^2+1}{x+1}$, we would write "$\lim_{x\to 0} \frac{x^2+1}{x+1} = \lim_{x\to 0} \frac{2x}{1} = 0$" but this is <u>wrong</u> since we can just plug in $x=0$ to see that $\lim_{x\to 0} \frac{x^2+1}{x+1} = \frac{0^2+1}{0+1} = \frac{1}{1} = \underline{\underline{1}}.$

Sometimes limits look like "$0\cdot\infty$". These are really indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ "in disguise"!

E.g. Looking at $\lim_{x\to\infty} x\cdot e^{-x}$ we have $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to\infty} e^{-x} = 0$.

We can rewrite e^{-x} as $\frac{1}{e^x}$ to then use L'Hôpital:

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{d/dx(x)}{d/dx(e^x)} = \lim_{x\to\infty} \frac{1}{e^x} = \underline{\underline{0}}.$$

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Anti-derivatives §4.9

Whenever we have some "operation" in mathematics, it is useful to think about "undoing" this operation: e.g. we discussed how inverse functions (like $\ln(x)$) undo the original functions (like e^x).

Differentiation is an important operation, and its "inverse" is called anti-differentiation.

Def'n We say that $F(x)$ is an anti-derivative of $f(x)$ if $F'(x) = f(x)$ (on some interval).

E.g. $F(x) = x^2$ is an anti-derivative of $f(x) = 2x$ since $d/dx\,(x^2) = 2x$.

NOTE: There are multiple anti-derivatives of $f(x)$:

E.g. x^2+1 is another anti-derivative of $2x$.

But... Thm If $F(x)$ is one particular anti-derivative of $f(x)$, then the general anti-derivative is $F(x) + C$ for all constants $C \in \mathbb{R}$.

Pf: We explained this before, using Mean Value Thm. ∎

The $+C$ part is important, but this theorem tells us it is enough to know one anti-derivative of $f(x)$ in order to understand them all.

Unfortunately, it can be pretty hard to find anti-derivatives...

e.g. for $f(x) = e^{x^2}$, we know how to compute its derivative, but there is no simple way to compute its anti-derivative.

But... we will still learn how to compute certain anti-derivatives

Let's start with something easy:

Theorem. • If $F(x)$ is anti-derivative of $f(x)$, then $c \cdot F(x)$ is a.-d. of $c \cdot f(x)$ for all $c \in \mathbb{R}$.

• If $F(x)$ is a.-d. of $f(x)$ and $G(x)$ is a.-d. of $g(x)$, then $F(x) + G(x)$ is a.-d. of $f(x) + g(x)$.

Pf: These follow from linearity of derivative:

$$d/dx\,(c \cdot F(x) + d \cdot G(x)) = c \cdot F'(x) + d \cdot G'(x). \quad \blacksquare$$

But what about something like $f(x) = x^n$?

How to find an anti-derivative of x^n?

Notice that $d/dx\,(x^{n+1}) = (n+1) \cdot x^n$, almost what we want, just need to divide by $n+1$. (But with $n = -1$, this doesn't work!)

Some common antiderivatives:

$f(x)$	(particular) anti-derivative $F(x)$
$x^n \quad (n \neq -1)$	$\frac{1}{n+1} \cdot x^{n+1}$
$1/x$	$\ln(x)$
e^x	e^x
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$

notice how the $-$ sign is "backwards" from the derivative.

This table gives us many anti-derivatives, but to deal with more complicated $f(x)$, we'll learn more techniques! (like $f(x) = \cos^2(x)$)