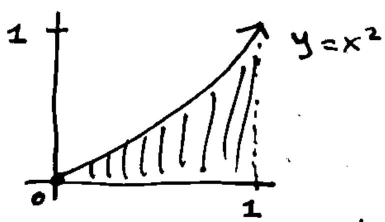
11/13

Area under a curve: § 5.1

On the 1^{st} day of class, we briefly discussed two problems that calculus solves: the tangent to a curve, and the area under a curve.

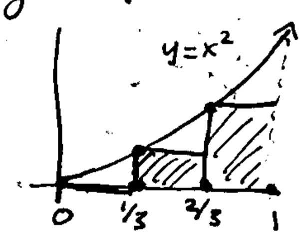
We've spent many weeks discussing the tangent and its relation to the derivative. We end the semester discussing area under a curve and the integral.

Let $f(x) = x^2$ and consider curve $y = f(x)$. What's the area between this curve and the x-axis, for $0 \le x \le 1$?

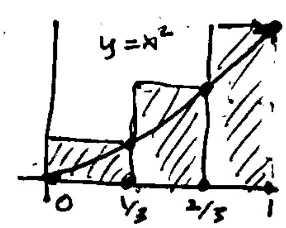


$\Leftarrow A$ = area of shaded region

In geometry we learn formulas for areas of shapes like triangles, rectangles, circles, ... but this shape is different. However, we could approximate the area A by using shapes like rectangles which are easy to work with.



L_3 = area of $\leftarrow$ ▨



R_3 = area of $\leftarrow$ ▨

On the left we drew 3 rectangles of width $1/3$ where the left vertex of the top of each rectangle touches $y = f(x)$, and on the right we drew 3 rectangles of width $1/3$ where the right vertex of the top of each rect. touches $y = f(x)$.

We see that $L_3 < A < R_3$.

width of 1st rectangle ↓ — height of 1st rect. — width of 2nd rect. — height of 2nd rect.

We can compute $L_3 = (\frac{1}{3})\cdot 0^2 + (\frac{1}{3})(\frac{1}{3})^2 + (\frac{1}{3})(\frac{2}{3})^2 = \frac{5}{27}$

and $R_3 = (\frac{1}{3})(\frac{1}{3})^2 + (\frac{1}{3})(\frac{2}{3})^2 + (\frac{1}{3})\,1^2 = \frac{14}{27}$

So that $0.1851\ldots = \frac{5}{27} < A < \frac{14}{27} = 0.5185\ldots$

If we let L_n and R_n denote the analagous areas of rectangles but where we use n rectangles of width $1/n$ (touching curve at left and right top vertices, resp.) then we always have $L_n < A < R_n$ and <u>larger values of n</u> give better <u>approximations</u>!

<u>e.g.</u> $n=10 \Rightarrow 0.285\ldots < A < 0.385\ldots$

$n=100 \Rightarrow 0.328\ldots < A < 0.338\ldots$

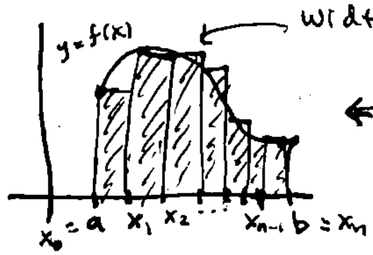
$n=1000 \Rightarrow 0.332\ldots < A < 0.333\ldots$

It looks like the bounds are <u>converging</u> to $1/3 = 0.\overline{3}$.

This is <u>true</u>! Suggests we can define area under curve as a <u>limit</u>:

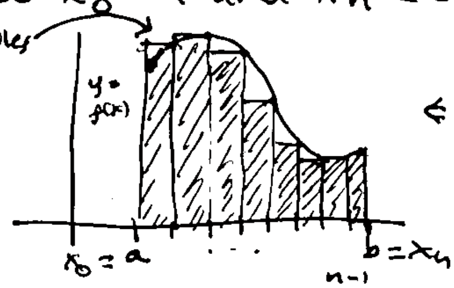
<u>Def'n</u> Let $f(x)$ be defined on a closed interval $[a,b]$.

Fix n, and let $\Delta x = \frac{b-a}{n}$, and let $x_i = a + i\cdot\Delta x$ for all $i = 0,1,2,\ldots,n$ (so $x_0 = a$ and $x_n = b$).



width of rectangles $= \Delta n$

← L_n



← R_n

Σ is shorthand for sum

Let $L_n = \Delta x\cdot f(x_0) + \Delta x\cdot f(x_1) + \cdots + \Delta x\, f(x_{n-1}) = \sum_{i=0}^{n-1} \Delta x\cdot f(x_i)$

and $R_n = \Delta x\cdot f(x_1) + \Delta x\cdot f(x_2) + \cdots + \Delta x\, f(x_n) = \sum_{i=1}^{n} \Delta x\cdot f(x_i)$

Then, as long as $f(x)$ is <u>continuous</u>, the limits of the areas $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} R_n$ exist and are equal, so we define

A = <u>area under the curve</u> $= \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$.

E.g.: Let us return to $f(x) = x^2$ defined on $[0,1]$.

Then $R_n = \frac{1}{n} \cdot f(\frac{1}{n}) + \frac{1}{n} \cdot f(\frac{2}{n}) + \cdots + \frac{1}{n} f(\frac{n}{n})$

$$= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right).$$

<u>Proposition</u>. $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

E.g. $1^2 = 1 = \frac{1(1+1)(2+1)}{6}$, $1^2 + 2^2 = 5 = \frac{2(2+1)(4+1)}{6}$, ...

<u>Proof</u>: This can be proved by <u>mathematical induction</u>.
Maybe you have seen the similar formula:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

The n^2 one is slightly more complicated, but basically same. $\boxtimes$

So $R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6n^3}$

Thus $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{2}{6} = \frac{1}{3}$.

This definition of area under the curve in terms of limits of rectangle sums is conceptually clear, but difficult to compute with: we have to find formulas like $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

One of the main insights of <u>calculus</u> is that there is another way to find these areas using <u>anti-derivatives</u> of functions, which is much more computationally easy!

11/14

The Definite Integral § 5.2

Area under the curve is so important that we give it a special name and notation.

Def'n Let $f(x)$ be a continuous function defined on $[a,b]$. The (definite) integral of $f(x)$ from a to b is

$$\int_a^b f(x)\,dx = \text{area under curve } y=f(x) \text{ from } x=a \text{ to } x=b.$$

More precisely, fix n and let $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\cdot\Delta x$ for $i = 0,1,\dots,n$. Choose a point $x_i^* \in [x_{i-1}, x_i]$ for each $i = 1,2,\dots,n$. Then define

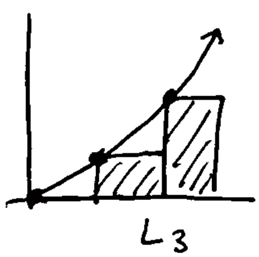
$$A_n = \sum_{i=1}^{n} \Delta x \cdot f(x_i^*)$$

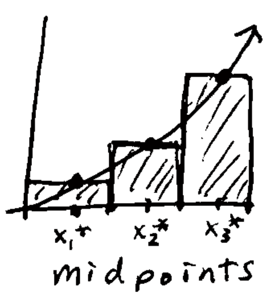
and finally $\int_a^b f(x)\,dx = \lim_{n\to\infty} A_n$.

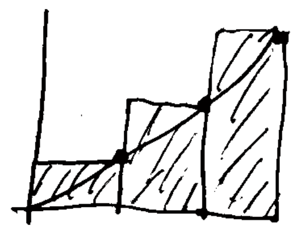
Note: If we choose $x_i^* = x_{i-1}$ for all i, then $A_n = L_n$. If we choose $x_i^* = x_i$ for all i, then $A_n = R_n$.

But no matter which point we choose to determine the height of the thin rectangles in our approximation of the area under the curve, in the limit all give the same value.

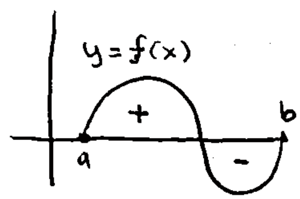
However, for some fixed n, the approximations will be different, and often the best choice is to use midpoints $x_i^* = \frac{x_{i-1}+x_i}{2}$.







For $f(x)$ always above x-axis, $\int_a^b f(x)\,dx$ really is the area under the curve, but for $f(x)$ that goes below the x-axis, we have to subtract that area:



$$\int_a^b f(x)\,dx = +\left(\text{area above x-axis and below } y=f(x)\right) - \left(\text{area below x-axis and above } y=f(x)\right)$$

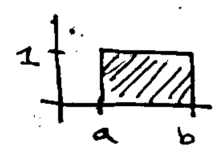
Some more properties of the integral:

Thm $\int_a^b (c\cdot f(x) + d\cdot g(x))\,dx = c\cdot\int_a^b f(x)\,dx + d\cdot\int_a^b g(x)\,dx$ for $c,d \in \mathbb{R}$ constants.

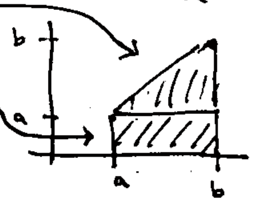
In other words, the integral is linear (just like the derivative).

Pf: $\sum_{i=1}^n \Delta x(cf(x_i) + dg(x_i)) = c\cdot\sum_{i=1}^n f(x_i) + d\cdot\sum_{i=1} g(x_i)$. ∎

E.g. $\int_a^b 1\,dx = (b-a)$ since just have a rectangle →



$$\int_a^b x\,dx = \overbrace{a\cdot(b-a)}^{\text{rectangle}} + \overbrace{\tfrac{1}{2}(b-a)(b-a)}^{\text{triangle}}$$
$$= \tfrac{1}{2}(a+b)(b-a) = \tfrac{1}{2}(b^2-a^2)$$



So that $\int_a^b (mx+c)\,dx = \frac{m}{2}(b^2-a^2) + c(b-a)$ and we now know the integral of any linear function.

Even though we only defined $\int_a^b f(x)\,dx$ when $a \le b$ it also makes sense to let $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, i.e. swapping endpoints of integral negates it.

Notice in particular that $\int_a^a f(x)\,dx = 0$.

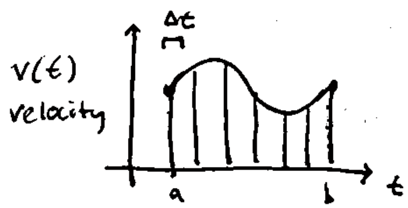
Also... Proposition For any $c \in [a,b]$, we have

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

Pf: Picture:  ← split the area into two pieces ▨

Position from velocity: We explained how the derivative (slope of tangent) lets us compute the velocity $v(t)$ of a car at time t if all we know is its position function $p(t)$. The integral does the opposite!

Specifically, suppose we know $v(t)$, velocity of a car as function of time t, on some interval $[a,b]$.



If $v(t)$ were constantly = fixed v, then the distance the car travels from time a to time b would just be $= v \cdot \underbrace{(b-a)}_{\text{elapsed time}}$

But since the velocity is changing, we need to measure it at multiple times in the interval $[a,b]$. We can approximate the distance traveled by letting $\Delta t = \frac{b-a}{n}$ and $t_i = a + i \cdot \Delta t$ for $i = 0, 1, \ldots, n$.

Then distance traveled $\approx \sum_{i=1}^{n} \Delta t \cdot v(t_i)$

since on each short time interval $[t_{i-1}, t_i]$ the velocity is approximately constant.

And in the limit, we have exactly that:

$$\begin{array}{c}\text{distance car} \\ \text{travels in} \\ \text{time } a \text{ to time } b\end{array} = \int_a^b v(t)\,dt, \text{ the integral!}$$

11/15

The Fundamental Theorem of Calculus §5.3

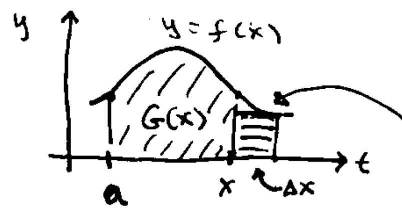
The following theorem gives a way to compute integrals:

Theorem Let $f(x)$ be a continuous function.

1) Define the function $G(x) = \int_a^x f(t)\,dt$ (for a fixed $a \in \mathbb{R}$).
   Then $G'(x) = f(x)$.
2) Suppose that $F(x)$ is any anti-derivative of $f(x)$.
   Then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Pf: This is just a proof sketch, see book for details.

For 1) The function $G(x)$ computes area under the curve $y = f(t)$ for $t = a$ to $t = x$:



If we increase x by Δx, then how does $G(x)$ change? Well, since $f(x)$ is continuous, we roughly add $\Delta x \cdot f(x)$ to $G(x)$.

Thus, $\Delta G \approx \Delta x \cdot f(x)$, i.e., $f(x) \approx \frac{\Delta G}{\Delta x}$.

As $\Delta x \to 0$, we get exactly that $\frac{dG}{dx} = f(x)$. ✓

For 2): We know from 1) that $G(x)$ is one anti-derivative of $f(x)$ (since $G'(x) = f(x)$).

So there is some constant $c \in \mathbb{R}$ such that $G(x) = F(x) + c$.

Now, $G(a) = \int_a^a f(x)\,dx = 0$, so $c = -F(a)$.

Thus, $\int_a^b f(x)\,dx = G(b) = F(b) - F(a)$. ✓

▨

For us the point of the Fund. Thm. of Calculus is that it lets us evaluate integrals by computing anti-derivatives.

E.g.: We saw before that $\int_0^1 x^2 = 1/3$.

Let's do this again, faster. Recall that $F(x) = 1/3\, x^3$ is one anti-derivative of $f(x) = x^2$ since $F'(x) = f(x)$.

Thus, by F.T.C., $\int_0^1 x^2 dx = F(1) - F(0) = 1/3 (1)^3 - 1/3 (0)^3 = \underline{\underline{1/3}}$.

Since we so often want to compute $F(b) - F(a)$, we use the shorthand notation $F(x)\Big]_a^b = F(b) - F(a)$.

Thus, F.T.C. says that $\int_a^b f(x)\,dx = F(x)\Big]_a^b$.

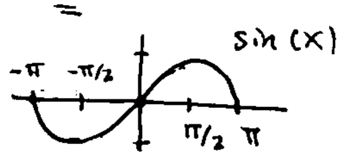
E.g. To compute $\int_1^2 e^x dx$, we recall that e^x is the anti-derivative of e^x, so that

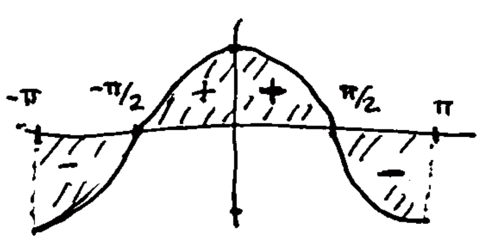
$$\int_1^2 e^x dx = e^x\Big]_1^2 = e^2 - e^1 = e(e-1).$$

E.g.: $\sin(x)$ is an anti-derivative of $\cos(x)$, so

$$\int_{-\pi}^{\pi} \cos(x)\,dx = \sin(x)\Big]_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi)$$
$$= 0 - 0$$
$$= \underline{0}.$$

This makes sense, since:





← areas above and below curve $y = \cos(x)$ from $x = -\pi$ to $x = \pi$ cancel out, leaving $\underline{0}$ overall.

11/20

Indefinite Integrals §5.4

We want better notation for anti-derivatives. This will come from the so-called indefinite integral

Def'n We write $\boxed{\int f(x)\,dx = F(x)}$ to mean that $F'(x) = f(x)$. The expression "$\int f(x)\,dx$" is called an indefinite integral.

Note: Do not confuse definite and indefinite integrals. The definite integral $\int_a^b f(x)\,dx$ is a number: it is the area under the curve $y=f(x)$ from $x=a$ to $x=b$. The indefinite integral $\int f(x)\,dx$ is a function: it is the anti-derivative of $f(x)$.

E.g. $\int_0^1 x^2\,dx = 1/3$, as we have seen.

But $\int x^2\,dx = \frac{1}{3}x^3 + C$ (for any $C \in \mathbb{R}$).

Table of indefinite integrals we know so far:

- $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ (for any $n \neq -1$)
- $\int e^x\,dx = e^x + C$
- $\int \frac{1}{x}\,dx = \ln(x) + C$
- $\int \sin(x)\,dx = -\cos(x) + C$
- $\int \cos(x)\,dx = \sin(x) + C$

(here $C \in \mathbb{R}$ is any constant)

With this indefinite integral notation, we can restate the Fundamental Theorem of Calculus as:

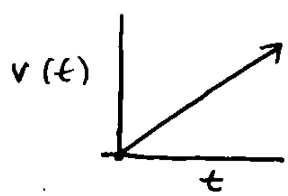
$$\underbrace{\int_a^b f(x)\,dx}_{\text{definite integral}} = \underbrace{\int f(x)\,dx\Big]_a^b}_{\text{indefinite integral } F(x) \text{ evaluated: } F(b)-F(a)}$$

Net Change: Another way to think of FTC:

$$\int_a^b F'(x)\,dx = F(b) - F(a) \quad \text{is}$$

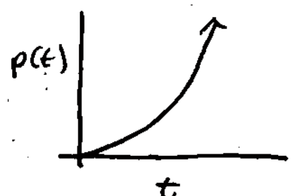
"the integral of the (instantaneous) rate of change is the net change (over some time interval)."

Eg: 1) If $p(t)$ is the position of a car (on a 1-D road) at time t, we have seen that $p'(t) = v(t)$ is the velocity, a.k.a., speed of the car. Thus $\int_a^b v(t)\,dt = \int_a^b p'(t)\,dt = P(b) - P(a)$ means that the integral of velocity (from time a to b) is the net displacement.



Velocity of car (w/ constant acceleration)

$\Rightarrow$



$p(t) = \int v(t)\,dt$

position function is integral of velocity.

2) In biology, if $n(t)$ is the number of organisms in some population at time t, then dn/dt is the rate of growth of the population. Hence $\int_a^b dn/dt\,dt = n(b) - n(a)$ is the net population growth from time a to time b.

3) In economics, if $P(x)$ is the profit from selling x units of some product, then dP/dx is the marginal profit. The FTC says the integral of marginal profit = total profit.

11/22

Integration by Substitution § 5.5

There are many integrals like $\int x \cdot \cos(x^2+1)\,dx$ where the rules we know for integration so far do not apply. One more advanced technique for integration is called integration by substitution or "u-substitution" for short.

Theorem If f, g are two differentiable functions then
$$\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C.$$

Proof: By chain rule, $\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x)$. ☒

How to use this theorem in practice? Let's see...

Eg: We want to compute $\int x \cdot \cos(x^2+1)\,dx$.

Let's set $\boxed{u = x^2+1}$ (think $u = g(x)$ is a function of x).

Then $\frac{du}{dx} = 2x$, or in differential notation $\boxed{du = 2x\,dx}$.

This means
$$\int x \cdot \cos(x^2+1)\,dx = \int \underbrace{\cos(x^2+1) \cdot \tfrac{1}{2}} \cdot \underbrace{2x\,dx}$$
$$= \int \tfrac{1}{2}\cos(u) \cdot du$$
$$= \tfrac{1}{2}\int \cos(u)\,du = \tfrac{1}{2}\sin(u) + C$$
$$\boxed{= \tfrac{1}{2}\sin(x^2+1) + C}$$

This is how the u-substitution technique works!

The above theorem says we can treat the dx (and the du) in an integral like the dx, du in $\frac{dx}{du}$.

But... we cannot mix functions of u with dx: must only integrate things like $\int h(u)\,du$, Not $\int h(u)\,dx$.

The steps to use u-substitution are:

- decide what $u = g(x)$ should be
- figure out what du is in terms of dx
- convert $\int f(x)\,dx$ to $\int h(u)\,du$ by making the appropriate substitutions
- hopefully $\int h(u)\,du = H(u)$ is an integral (anti-derivative) you already know how to do
- convert from u back to x: write $H(u) = F(x)$ ✓

Let's do some more examples:

E.g. $\int x^2 \cdot e^{4x^3+2}\,dx$.

We see "$4x^3+2$" inside the exponential, so a guess is that a good choice for u might be

$$u = 4x^3 + 2 \Rightarrow du = 12x^2\,dx$$

Since x^2 is there in the integrand, we're in luck!

$$\int x^2 e^{4x^3+2}\,dx = \int \underbrace{\tfrac{1}{12} e^{4x^3+2}} \cdot \underbrace{12x^2\,dx}$$

$$= \int \tfrac{1}{12} e^{u} \cdot du$$

$$= \tfrac{1}{12} e^u + C$$

we know how to integrate e^u

$$= \boxed{\tfrac{1}{12} e^{4x^3+2} + C}.$$

since $u = 4x^3+2$ this is how we go back to a function of x

If we're ever in doubt of our answer, we can always double-check it by differentiating:

$$d/dx \left(\tfrac{1}{12} e^{4x^3+2}\right) = \tfrac{1}{12} e^{4x^3+2} \cdot 12x^2 = x^2 e^{4x^3+2}. \checkmark$$

chain rule factor

E.g.: $\int 2x\sqrt{3x^2+1}\,dx$

Good choice of u is $u = 3x^2+1 \Rightarrow du = 6x\,dx$

$$\int 2x\sqrt{3x^2+1}\,dx = \int \underbrace{\tfrac{1}{3}\sqrt{3x^2+1}}\,\underbrace{6x\,dx}$$

$$= \int \tfrac{1}{3}\sqrt{u}\,du = \tfrac{1}{3}\int u^{1/2}\,du$$

recall $\int u^{1/2}\,du = \frac{2}{3}u^{3/2}$ by rule for anti-derivative of u^n

$$= \tfrac{1}{3}\cdot\tfrac{2}{3}u^{3/2} + C = \boxed{\tfrac{2}{9}(3x^2+1)^{3/2} + C}.$$

go back to x

E.g.: $\int \sin(x)\cos(x)\,dx$

This one is a little trickier... no polynomial expression involving x appears.

Instead, try $u = \sin(x) \Rightarrow du = \cos(x)\,dx$

This is good since both sin(x) and cos(x) appear!

So, $\int \sin(x)\cos(x)\,dx = \int u\,du$

$$= \tfrac{1}{2}u^2 + C$$

$$= \boxed{\tfrac{1}{2}\sin(x)^2 + C}.$$

You could also try $u = \cos(x)$ here... what would that give?

As you can see from these examples, using the u-substitution technique is a bit of an art since you have to find a clever choice of

$$\boxed{u = g(x)}$$

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