Math 156 | Calculus I | Final Exam A

12/6/2022

SHOW ALL WORK. Justify your answers!
Simplify your answers. Give exact answers whenever possible.

PART I : Answer all 4 of the following questions worth 24 points each.

1. Do the following.
   (a) State the limit definition of the derivative of f at x.
   (b) Use the limit definition in part (a) to show that the derivative of $3x^2 + 7x$ is $6x + 7$.

2. Find the slope of the tangent line to the function $f(x) = \dfrac{x^2}{x+3}$ at the point $(1, \frac{1}{4})$.

3. The interval $[1, 9]$ is partitioned into n subintervals $[x_{k-1}, x_k]$ for $k = 1, ..., n$, each of width Δx. Choose any x_k^* such that $x_{k-1} \le x_k^* \le x_k$. Let the function f be continuous over $[1, 9]$. Do the following.
   (a) State the limit definition of $\displaystyle\int_1^9 f(x)\,dx$.
   (b) Estimate the integral in (a) if $f(x) = x^2$ using a Riemann sum with $n = 4$ subintervals of equal width and sample points $x_k^* = x_k$ for $k = 1, 2, 3, 4$.
   (c) Sketch $f(x) = x^2$ and the rectangles whose area is the Reimann sum in (b). Use this sketch to explain why the sum in (b) overestimates the value of the integral in (a) when $f(x) = x^2$.

4. Evaluate each of the following integrals.
   (a) $\displaystyle\int \frac{x^3 - x}{x^2}\,dx$ (b) $\displaystyle\int_1^4 \left(\frac{2}{\sqrt{x}} + 6x\right) dx$

PART II : Answer any 8 of the following questions worth 18 points each.

5. Let f be the function defined by
$$f(x) = \begin{cases} -3x^2 + 1 & , \ x < 2 \\ x - 13 & , \ x \ge 2 \end{cases}.$$
   (a) Evaluate $\displaystyle\lim_{x\to 2^-} f(x)$, $\displaystyle\lim_{x\to 2^+} f(x)$, and $\displaystyle\lim_{x\to 2} f(x)$.
   (b) Is f continuous at 2? Use the definition of continuity to justify your answer.
   (c) If f differentiable at 2? Show the work that leads to your conclusion.

6. Let $f(x) = \dfrac{-5x}{e^x - e}$. Do the following.
   (a) Evaluate the limits : $\displaystyle\lim_{x\to 1^+} f(x)$ and $\displaystyle\lim_{x\to\infty} f(x)$.
   (b) State the equation(s) of any horizontal and/or vertical asymptote(s) to the graph $y = f(x)$. Justify each answer with a limit statement.

(continued on the next page)

7. Find $\frac{dy}{dx}$ for each of the following.

$(a)\quad y = \sin^3 x^4 \qquad\qquad (b)\quad y^6 + xy = 7$

8. Do the following.

   (a) Determine the linearization of $f(x) = \ln x$ about the number e.

   (b) Using your answer from part (a), along with 2.718 as an approximation for e, approximate $\ln 2$ to 3 decimal places.

9. An 8 foot ladder is leaning against a wall that is perpendicular to the level floor. Let θ be the angle between the bottom of the ladder and the floor. Do the following.

   (a) If the bottom of the ladder is being pushed toward the wall at the constant rate of 3 feet per minute, how fast is θ increasing when the bottom of the ladder is 2 feet from the wall ?

   (b) Show that the area of the triangle formed by the ladder, the wall and the floor is $A = 16 \sin 2\theta$. You may use the identity $\sin 2\theta = 2\cos\theta \sin\theta$.

   (c) Use calculus to find the largest possible area of the triangle in (b).

10. Let $f(x) = (x^2 - 4)^3$. Do the following.

    (a) Find the maximum value and the minimum value of f on the interval $[-1, 3]$.

    (b) For what value of c such that $-1 \le c \le 3$ does f attain its maximum value ?

11. Let $f(x) = k\ln(x^2 + 3) + 5$, where $k < 0$. Do the following.

    (a) Find the critical numbers of f, and make a sign chart for $f'(x)$.

    (b) Find the open interval(s) on which f is increasing and the open interval(s) on which f is decreasing. Justify each answer.

    (c) Find the value of x where f has a local extreme value, and classify this extrema as a local maximum or minimum. Justify your answer.

12. The velocity of a particle traveling along a straight line is given by $v(t) = 2(t^2 - 2t - 3)$ for $2 \le t \le 4$.

    (a) Find the acceleration of the particle at the time when the particle is at rest.

    (b) Find the total distance traveled by the particle over the time interval $2 \le t \le 4$.

13. If $F(x) = \displaystyle\int_2^x \sqrt{4t^2 + 1}\, dt$, find $F(2), F'(2)$, and $F''(2)$.

14. Integrate the following.

$(a)\quad \displaystyle\int_1^2 (x-1)e^{(x-1)^2}\, dx \qquad\qquad (b)\quad \displaystyle\int \frac{\sin(\ln x)}{x}\, dx$

(end of exam)