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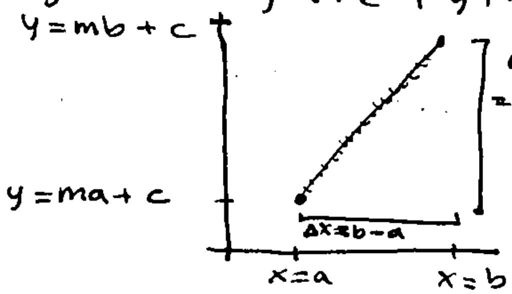
Arc lengths of curves § 8.1

Having studied techniques for integration, we return to applications of integrals. We've already used integrals to compute areas (2D measures), and volumes (3D measures), what about lengths (1D measures)?

Suppose we have a curve $y = f(x)$ from $x=a$ to $x=b$: what is the length of this curve? Of course, if the curve were a line $y = mx + c$ we could compute its length using the Pythagorean Theorem:



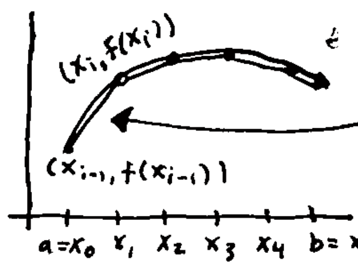
$\Delta y = (mb+c) - (ma+c) = m(b-a)$

$\Rightarrow$ length of line segment

$$= \sqrt{\Delta y^2 + \Delta x^2}$$
$$= \sqrt{m^2(b-a)^2 + (b-a)^2}$$
$$= (b-a)\sqrt{m^2+1}$$

Notice: length depends on slope of line

But what if $y = f(x)$ is not a line? As usual, we break it into smaller parts where we treat it as approximately linear:



- break $[a,b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ of width $\Delta x = \frac{b-a}{n}$
- length between $(x_{i-1}, f(x_{i-1}))$ & $(x_i, f(x_i))$ is $\sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$

$$= \sqrt{\Delta x^2 + \Delta y_i^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\,\Delta x$$

where $\Delta y_i = f(x_i) - f(x_{i-1})$

Thus, length of $y = f(x)$ from $x=a$ to $x=b$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\,\Delta x$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \boxed{\int_a^b \sqrt{1 + (f'(x))^2}\,dx}$$

In limit, $\frac{\Delta y_i}{\Delta x}$ becomes the derivative $\frac{dy}{dx}$

E.g. If $f(x) = mx + c$ is a line, then $f'(x) = m$ ✓

So length from $x=a$ to $x=b$ $= \int_a^b \sqrt{1+(f'(x))^2}\,dx = \int_a^b \sqrt{1+m^2}\,dx = (b-a)\sqrt{1+m^2}$

E.g. Consider the curve $y = x^{3/2}$ from $x=0$ to $x=1$.

Length $= \int_0^1 \sqrt{1+(\frac{d}{dx} x^{3/2})^2}\,dx = \int_0^1 \sqrt{1+(\frac{3}{2}x^{1/2})^2}\,dx$

$= \int_0^1 \sqrt{1+\frac{9}{4}x}\,dx$ ← can solve w/ a u-sub.

① indef. integral: $\int \sqrt{1+9/4\,x}\,dx = \int \sqrt{u}\ 4/9\,du = \frac{4}{9}\cdot\frac{2}{3}x^{3/2}$

$u = 1 + 9/4\,x$ $\qquad = 8/27\,(1+9/4\,x)^{3/2}$

$du = 9/4\,dx$

② plug in: $\int_0^1 \sqrt{1+9/4\,x}\,dx = \left[8/27\,(1+9/4\,x)^{3/2}\right]_0^1 = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2} - 1\right).$

E.g. Even for curve $y = x^2$ from $x=0$ to $x=1$, integral is nasty:

Length $= \int_0^1 \sqrt{1+(\frac{d}{dx}x^2)^2}\,dx = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx$

① indef. integral: $\int \sqrt{1+4x^2}\,dx$ good idea: trig sub! $x = \frac{1}{2}\tan\theta$

$dx = \frac{1}{2}\sec^2\theta\,d\theta$

$= \int \sqrt{1+\tan^2\theta}\ \frac{1}{2}\sec^2\theta\,d\theta = \frac{1}{2}\int \sec^3\theta\,d\theta$

But... $\int \sec^3\theta\,d\theta$ is not easy! Int. by parts helps, but even then you still need to know $\int \sec\theta\,d\theta = \ln(\sec\theta + \tan\theta)$.

E.g. Sometimes $(1+(f'(x))^2)$ has a square root:

If $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ then $f'(x) = \frac{1}{2}x - \frac{1}{2x} = \frac{x^2-1}{2x}$

So $1+(f'(x))^2 = 1 + \left(\frac{x^2-1}{2x}\right)^2 = 1 + \frac{x^4-2x^2+1}{4x^2} = \frac{x^4+2x^2+1}{4x^2} = \frac{(x^2+1)^2}{(2x)^2}$.

Thus, $\int \sqrt{1+(f'(x))^2}\,dx = \int \sqrt{\frac{(x^2+1)^2}{(2x)^2}}\,dx = \int \frac{x^2+1}{2x}\,dx = \int \frac{x}{2}\,dx + \int \frac{1}{2x}\,dx$

$= \frac{x^2}{4} + \frac{1}{2}\ln(x) + C$, so we get...

$\int_1^e \sqrt{1+(f'(x))^2}\,dx = \left[\frac{x^2}{4} + \frac{1}{2}\ln(x)\right]_1^e = \frac{e^2}{4} + \frac{1}{4}$. ← length of $y=f(x)$ from $x=1$ to $x=e$.

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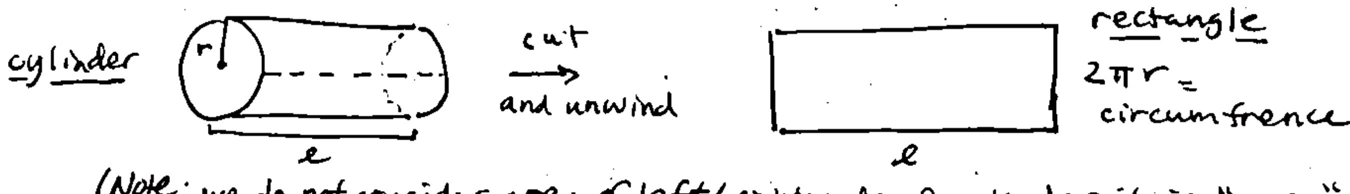
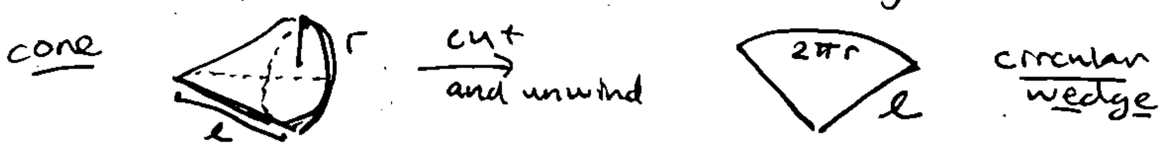
Area of Surface of Revolution §8.2

Intuitively, the surface area of a solid is the amount of "wrapping paper" you would need to wrap it.
As usual, we start our discussion of surface area with simple shapes. Consider a cylinder of length ℓ and radius r:



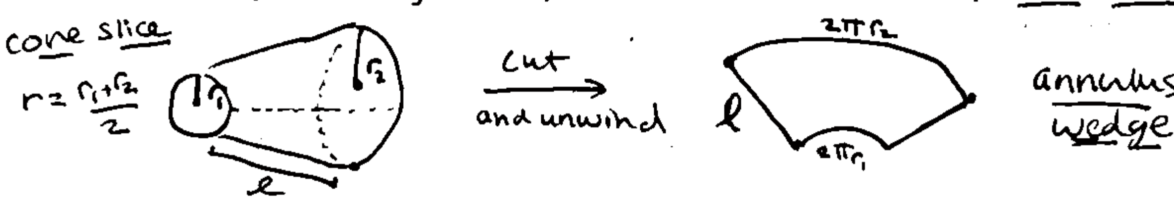
(Note: we do not consider area of left/right ends of cylinder, it is "open".)
By cutting the cylinder and unwinding it into a rectangle we see that it has surface area $= \boxed{2\pi r \cdot \ell}$

Similarly, if we take a cone of slant length ℓ and base radius r:



a simple calculation shows surface area $= \boxed{\pi r \ell}$

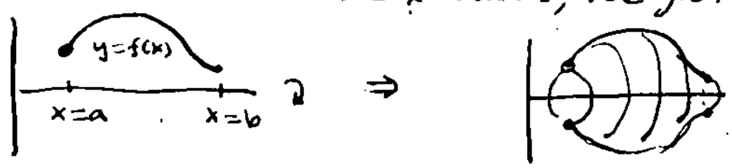
More generally still, if we consider a cone slice:



then its surface area $= \boxed{2\pi r \ell}$ where $\boxed{\ell = \text{slant length}}$ and $\boxed{r = (r_1+r_2)/2}$ is average of radii of the bases.

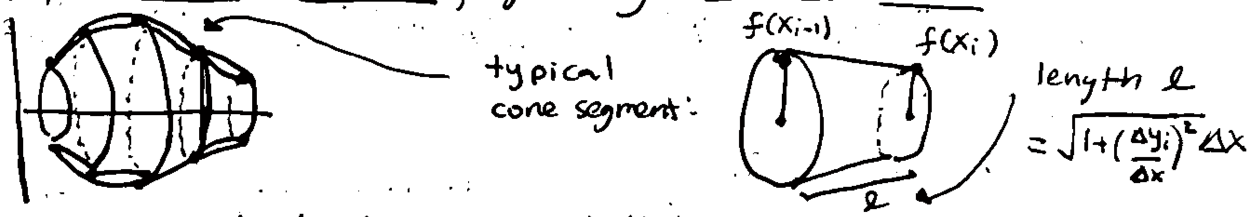
Cylinders, cones, and cone slices are all examples of surfaces of revolution, and we can use Calculus to obtain an integral formula for surface area of any surface of revolution...

Consider a curve $y=f(x)$ from $x=a$ to $x=b$. By rotating this curve around the x-axis, we get a surface of revolution:



So a surface of revolution is just the (lateral) boundary of the corresponding solid of revolution.

As usual, to find the area of a surface of revolution, we break the curve into short intervals where we approximate it by a linear function, giving cone segments:



We explained last class when talking about arc lengths that the slant length of the i^{th} cone segment $=\sqrt{1+(\frac{\Delta y_i}{\Delta x})^2}\cdot\Delta x$

Meanwhile, the circumfrence $=2\pi f(x_i^*)$ for some $x_i^* \in [x_{i-1}, x_i]$.

So the area of the i^{th} segment $=2\pi f(x_i^*)\cdot\sqrt{1+(\frac{\Delta y_i}{\Delta x})^2}\,\Delta x$.

and the total area of surface $\approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+(\frac{\Delta y_i}{\Delta x})^2}\,\Delta x$.

Taking the limit as $n\to\infty$, we get

Area of surface of revolution rotating $y=f(x)$ from $x=a$ to $x=b$ about x-axis $=\boxed{\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx}$

To remember this formula, think:

circumfrence $\times$ length

$2\pi f(x) \cdot \sqrt{1+(\frac{dy}{dx})^2}\,dx$

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E.g. Consider $y=\sqrt{x}$ from $x=0$ to $x=1$ rotated about x-axis.

Area of surface of revolution $= \int_0^1 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ where $f(x) = x^{1/2}$, $f'(x) = 1/2\, x^{-1/2}$

$$= \int_0^1 2\pi\sqrt{x}\sqrt{1+\left(\tfrac{1}{2}\tfrac{1}{\sqrt{x}}\right)^2}\,dx = \int_0^1 2\pi\sqrt{x}\sqrt{1+\tfrac{1}{4x}}\,dx$$

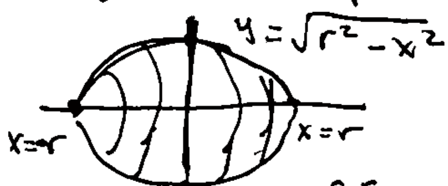
$$= 2\pi\int_0^1 \sqrt{x\cdot\left(1+\tfrac{1}{4x}\right)}\,dx = 2\pi\int_0^1\sqrt{x+\tfrac{1}{4}}\,dx$$

① Indef. integral: $\int\sqrt{x+\frac{1}{4}}\,dx = \int\sqrt{u}\,du = \frac{2}{3}u^{3/2}$

$u = x+\frac{1}{4}$, $du = dx$ $\qquad = 2/3\left(x+\frac{1}{4}\right)^{3/2}$

② Plug in FTC: $\Rightarrow 2\pi\int_0^1\sqrt{x+\frac{1}{4}}\,dx = 2\pi\left[2/3\left(x+\tfrac{1}{4}\right)^{3/2}\right]_0^1 = \frac{4\pi}{3}\left(\left(\tfrac{5}{4}\right)^{3/2} - \left(\tfrac{1}{4}\right)^{3/2}\right)$.

E.g. Let's compute the surface area of a sphere of radius r.



⇐ recall sphere is surface of revolution for $y=f(x)=\sqrt{r^2-x^2}$ for $x=-r$ to $x=r$

we compute $f'(x) = -2x\cdot\frac{1}{2}(r^2-x^2)^{-1/2} = \dfrac{-x}{\sqrt{r^2-x^2}}$

So area $= \int_{-r}^{r} 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$

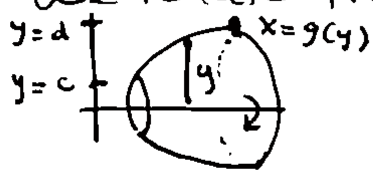
$$= \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx = \int_{-r}^{r}2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$= 2\pi\int_{-r}^{r}\sqrt{(r^2-x^2)\left(1+\frac{x^2}{r^2-x^2}\right)}\,dx = 2\pi\int_{-r}^{r}\sqrt{(r^2-x^2)+x^2}\,dx$$

$$= 2\pi\int_{-r}^{r}\sqrt{r^2}\,dx = 2\pi r\int_{-r}^{r}dx = \boxed{4\pi r^2}$$

Note: If we did $\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ here instead we would get $\int_a^b 2\pi r\,dx = 2\pi r(b-a)$, which gives the surface area of sphere segment from $x=a$ to $x=b$ (result of Archimedes!)

It is also possible to compute surface area by integrating w.r.t. $\underline{\underline{y}}$.
Suppose that $x = g(y)$ for $y = c$ to $y = d$, and we rotate this curve about the x-axis:



$\Leftarrow$ Same surface of revolution but given x in terms of y

A similar computation shows that

$$\text{area} = \boxed{\int_c^d \underbrace{2\pi y}_{\text{circumfrence}} \cdot \underbrace{\sqrt{1+(g'(y))^2}\,dy}_{\text{length}}}$$

Think: circumfrence × length

$$\sqrt{1+(dx/dy)^2}\,dy = \sqrt{1+(dy/dx)^2}\,dx$$

E.g. Consider curve $x = \frac{2}{3}y^{3/2}$ from $y=0$ to $y=3$.
Compute surface area of surface get by rotating about x-axis.
Since we already have x in terms of y, it is easiest here to use the y-integral formula:

$$\underline{\text{Area}} = \int_c^d 2\pi y \sqrt{1+(g'(y))^2}\,dy \quad \text{where } g(y) = 2/3\, y^{3/2} \text{ so } g'(y) = y^{1/2}$$

$$= \int_0^3 2\pi y \sqrt{1+(y^{1/2})^2}\,dy = \int_0^3 2\pi\, y\sqrt{1+y}\,dy$$

① indef. integral: $\int y\sqrt{1+y}\,dy = \int (u-1)\sqrt{u}\,du$

$u = 1+y \Rightarrow y = u-1$, $du = dy$

$$= \int u^{3/2} - u^{1/2}\,du$$
$$= 2/5\,u^{5/2} - 2/3\,u^{3/2} = 2/5(1+y)^{5/2} - 2/3(1+y)^{3/2}$$

② Plug in FTC $\Rightarrow 2\pi\int_0^3 y\sqrt{1+y}\,dy = 2\pi\left[2/5(1+y)^{5/2} - 2/3(1+y)^{3/2}\right]_0^3$

$$= 2\pi\left(\left(2/5\, 4^{5/2} - 2/3\, 4^{3/2}\right) - \left(2/5 - 2/3\right)\right) = \cdots = \frac{232\pi}{15}.$$

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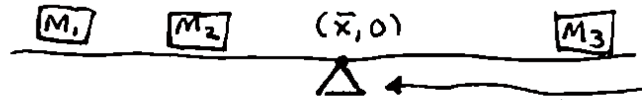
3/1 Center of Mass and Centroid §8.3

Suppose we have n objects $O_1, O_2, \ldots, O_n$ on a line, where O_i is located at $(x_i, 0)$ and has mass M_i:

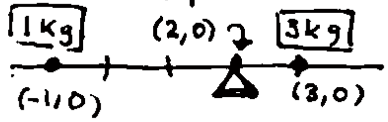


At what point should we place the fulcrum of a scale so that the objects will be perfectly balanced?
This point is called the center of mass and can be computed by formula

$$\boxed{\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}}$$

← think: weight each point by mass of object there

E.g. Suppose we have 1kg object at $(-1,0)$, 3kg object at $(3,0)$:



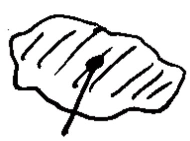
$$\bar{x} = \frac{(-1 \cdot 1 + 3 \cdot 3)}{1+3} = \frac{8}{4} = 2$$

so center of mass is at $(2,0)$

Similarly, if objects $O_1, \ldots, O_n$ are on the plane with O_i located at (x_i, y_i) and having mass m_i, we define the center of mass to be $(\bar{x}, \bar{y})$ where

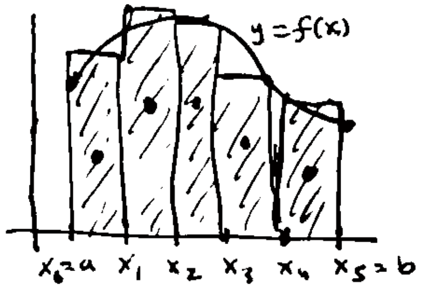
$$\bar{x} = \frac{\sum_i x_i m_i}{\sum_i m_i} \quad \text{and} \quad \bar{y} = \frac{\sum_i y_i m_i}{\sum_i m_i}$$

But what if instead of discrete point masses, we have a region of mass? Can still ask for the center of mass

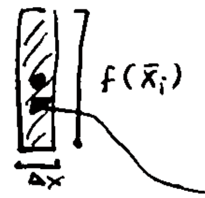


⟸ as the "balancing point" if we imagine the region as a "plate" balancing on a "stick".

For simplicity, assume the region has uniform density (e.g. 1kg/unit area), then center of mass is called the centroid. And suppose the region is the region below curve $y = f(x)$ from $x=a$ to $x=b$:



⇐ as usual, we let $\Delta x = \frac{b-a}{n}$ and set $x_i = a + i\Delta x$ for $i = 0, 1, \ldots, n$ and break region up into rectangles:



Let $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$ be mid point of $[x_{i-1}, x_i]$ so that i^{th} rectangle has width Δx and height $f(\bar{x}_i)$

Since the density is uniform (1kg/unit area):

mass of i^{th} rectangle = width × height $= f(\bar{x}_i)\,\Delta x = m_i$

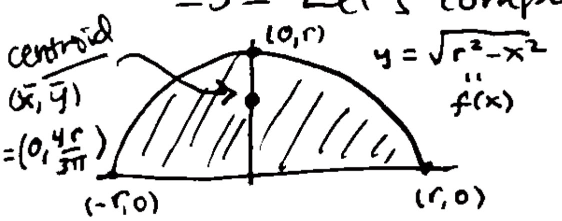
And the centroid of the rectangle is its middle: $\left(\bar{x}_i, \frac{f(\bar{x}_i)}{2}\right)$

So imagine we had point masses of mass $m_i = f(\bar{x}_i)\Delta x$ and at locations $\left(\bar{x}_i, \frac{f(\bar{x}_i)}{2}\right)$, then center of mass would be $\bar{x} \approx \frac{\sum_{i=1}^{n} \bar{x}_i f(\bar{x}_i)\Delta x}{\sum_i f(\bar{x}_i)\Delta x}$, $\bar{y} \approx \frac{\sum_{i=1}^{n} \frac{f(\bar{x}_i)}{2} f(\bar{x}_i)\Delta x}{\sum_i f(\bar{x}_i)\Delta x}$

Letting $n \to \infty$, we get that the centroid is $(\bar{x}, \bar{y})$ where

$$\boxed{\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx, \quad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}(f(x))^2\,dx \quad \text{with } A = \int_a^b f(x)\,dx = \text{area of region}}$$

E.g.: Let's compute the centroid of a semicircle of radius r:



Here we could compute area $A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$ but we already know from geometry that $A = \pi r^2/2$.

Similarly, we could compute $\bar{x} = \frac{1}{A}\int_{-r}^{r} x\sqrt{r^2 - x^2}\,dx$, but clear from symmetry that we must have $\bar{x} = 0$.

So we only need to compute $\bar{y} = \frac{1}{A}\int_{-r}^{r} \frac{1}{2}(f(x))^2\,dx = \frac{1}{\pi r^2/2}\int_{-r}^{r} \frac{1}{2}\left(\sqrt{r^2 - x^2}\right)^2 dx$

$$= \frac{1}{\pi r^2}\int_{-r}^{r} r^2 - x^2\,dx = \frac{1}{\pi r^2}\left[r^2 x - \frac{1}{3}x^3\right]_{-r}^{r}$$

$$= \frac{1}{\pi r^2}\left(\left(r^3 - \frac{1}{3}r^3\right) - \left(-r^3 + \frac{1}{3}r^3\right)\right) = \frac{1}{\pi r^2}\left(\frac{2}{3}r^3 + \frac{2}{3}r^3\right) = \boxed{\frac{4r}{3\pi}}$$ //