3/11

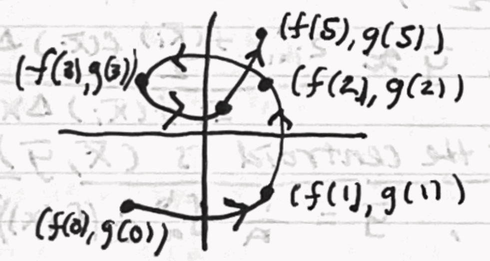
# Parametric Equations § 10.1

The $1^{st}$ half of the semester for Calc II focused on integration.
In $2^{nd}$ half we explore other topics, starting with Chapter 10 on parametric equations & polar coordinates.

Up until now we have considered curves of the form $y = f(x)$ (or more rarely, $f(x, y) = 0$).
A parametrized curve is defined by two equations:

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where $t$ is an auxilliary variable. Often we think of $t$ as time, so the curve describes motion of a particle where at time $t$ particle is at position $(f(t), g(t))$:



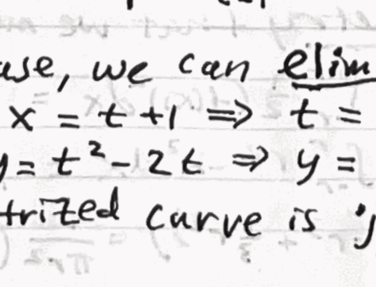

$\Leftarrow$ In this picture the arrows $\rightarrow$ show movement of particle over time

E.g. Consider parametrized curve $\boxed{x = t+1, \; y = t^2 - 2t}$

We can make a chart with various values of $t$:

| $t$ | $x$ | $y$ |
|---|---|---|
| -2 | -1 | 8 |
| -1 | 0 | 3 |
| 0 | 1 | 0 |
| 1 | 2 | -1 |
| 2 | 3 | 0 |
| 3 | 4 | 3 |

$\Rightarrow$




$\Leftarrow$ plot of points $(f(t), g(t))$ for $t = -1, 0, 1, \ldots, 4$ looks like a parabola

In this case, we can eliminate the variable $t$:

$$x = t+1 \Rightarrow t = x - 1$$

$$y = t^2 - 2t \Rightarrow y = (x-1)^2 - 2(x-1) = x^2 - 4x + 3$$

So this parametrized curve is just $\boxed{y = x^2 - 4x + 3}$

initial time $\Rightarrow$ initial point is $(f(0),g(0))$

terminal time $\Rightarrow$ terminal point is $(f(2\pi),g(2\pi))$

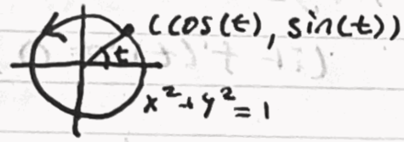
E.g. Consider the parametric curve:

$$x = \cos(t),\quad y = \sin(t) \quad \text{for} \quad 0 \le t \le 2\pi$$

How can we visualize this curve?

Notice that $x^2+y^2 = \cos^2(t) + \sin^2(t) = 1$,

so this parametrizes a circle $x^2+y^2=1$.




$\Leftarrow$ here $t$ = angle (in radians) of point $(\cos(t), \sin(t))$ on unit circle

E.g. What about $x = \cos(2t)$, $y = \sin(2t)$, $0 \le t \le 2\pi$?

Notice we still have $x^2+y^2 = \cos^2(2t) + \sin^2(2t) = 1$,

so the parametrized curve still traces a circle:

$\Leftarrow$ But now the parametrized curve traces the circle twice: once for $0 \le t \le \pi$ and once for $\pi \le t \le 2\pi$

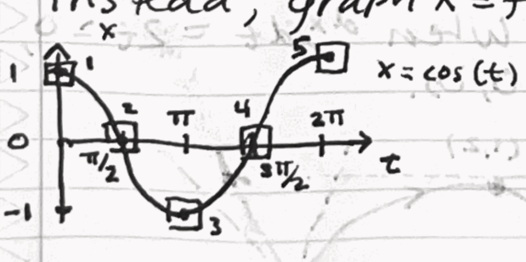
Can think of this particle as moving "faster" than the last one.

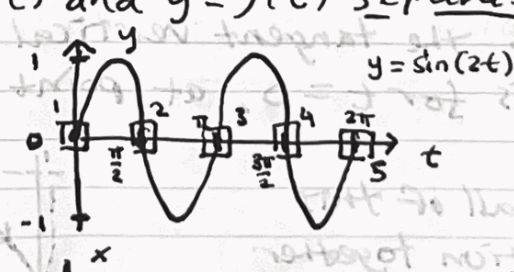
We see same curve can be parametrized in different ways!

E.g. Consider the curve $x = \cos(t)$, $y = \sin(2t)$.

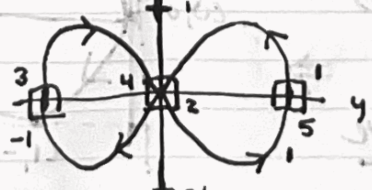
It's possible to eliminate $t$ to get $y^2 = 4x^2 - 4x^4$, but that equation is hard to visualize.

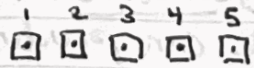
Instead, graph $x = f(t)$ and $y = g(t)$ separately:




Then combine into one picture showing $(f(t), g(t))$: $\Rightarrow$




$\Leftarrow$ are "snapshots" of the particle as it traces the curve

3/13

## Calculus with parametrized curves §10.2

Much of what we have done with curves of form $y=f(x)$ in calculus can also be done for parametrized curves:

Tangent vectors: Let $(x,y)=(f(t),g(t))$ be a curve. Then, at time $t$, the slope of tangent vector is given by:

$$\frac{dy}{dx} \underset{\text{chain rule}}{=} \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)} \quad (\text{if } f'(t)\neq 0)$$

If $dy/dt=0$ (and $dx/dt\neq 0$) $\Rightarrow$ horizontal tangent

If $dx/dt=0$ (and $dy/dt\neq 0$) $\Rightarrow$ vertical tangent

E.g. Consider curve $x=t^2$, $y=t^3-3t$.
First, notice that when $t=\pm\sqrt{3}$ we have

$$x=t^2=3 \quad \text{and} \quad y=t^3-3t=t(t^2-3)=0,$$

so curve passes thru $(3,0)$ at two times $t=\sqrt{3}$ and $t=-\sqrt{3}$.
We then compute that:

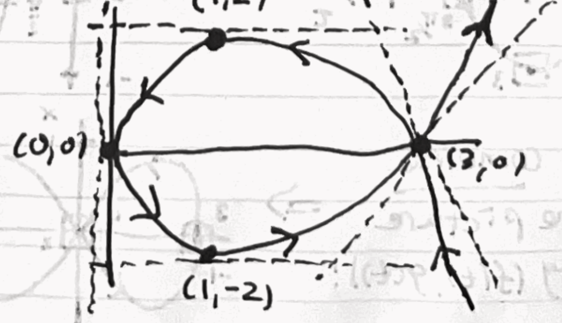
$$dy/dx = \frac{dy/dt}{dx/dt} = \frac{3t^2-3}{2t} \begin{cases} = -6/2\sqrt{3} = -\sqrt{3} \text{ at } t=-\sqrt{3} \\ = 6/2\sqrt{3} = \sqrt{3} \text{ at } t=\sqrt{3} \end{cases}$$

So two tangent lines, of slopes $\pm\sqrt{3}$, for curve at $(3,0)$.

When is the tangent horizontal? When $dy/dt=3t^2-3=0$ which is for $t=\pm 1$, at points $(1,2)$ and $(1,-2)$.

When is the tangent vertical? When $dx/dt=2t=0$, which is for $t=0$, at point $(0,0)$.

Putting all of this information together, we can produce a pretty good sketch of the curve $\Rightarrow$



**Arc lengths**: We saw several times how to find lengths of curves by breaking into line segments:

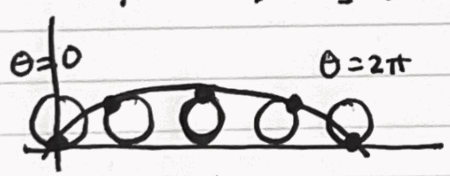
$\Leftarrow$ recall length of each small segment $= \sqrt{(\Delta x)^2 + (\Delta y)^2}$

For a parametrized curve $(x,y) = (f(t), g(t))$ with $\alpha \leq t \leq \beta$

we get that $\begin{matrix}\text{length} \\ \text{of curve}\end{matrix} = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt = \boxed{\int_\alpha^\beta \sqrt{f'(t)^2 + g'(t)^2}\, dt.}$

**Exercise**: Using parametrization $x = \cos(t), y = \sin(t)$, $0 \leq t \leq 2\pi$, show circumfrence of unit circle $= 2\pi$ using this formula.

**E.g.** The cycloid is the path a point on unit circle traces as the circle rolls:




$\Leftarrow$ think of this as an animation of a rolling circle, with point • marked where angle $\theta$ = "time"

The cycloid is parametrized by:

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta \quad \text{for } 0 \leq \theta \leq 2\pi$$

Q: What is the arclength of the cycloid?

A: We compute $\frac{dx}{d\theta} = 1 - \cos\theta$, $\frac{dy}{dx} = \sin\theta$ so that

$$\sqrt{(dx/dt)^2 + (dy/dt)^2} = \sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = \sqrt{2(1-\cos\theta)}$$

using trig identity $\frac{1}{2}(1 - \cos 2x) = \sin^2 x$

$$= \sqrt{4\sin^2(\theta/2)}$$
$$= 2\sin(\theta/2)$$

$\Rightarrow$ length of cycloid $= \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

$$= \int_0^{2\pi} 2\sin\left(\tfrac{\theta}{2}\right) d\theta = \left[-4\cos\left(\tfrac{\theta}{2}\right)\right]_0^{2\pi}$$

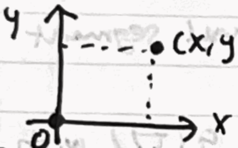
$$= ((-4 \cdot -1) - (-4 \cdot 1)) = \underline{8}.$$

//

3/15

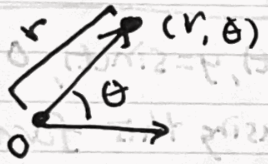
# Polar Coordinates §10.3

We are used to working with the "Cartesian" coordinate system where a point on the plane is represented by $(x, y)$



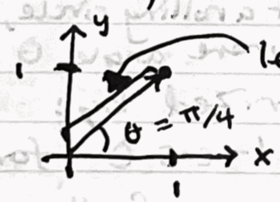

telling us how far to move along two orthogonal axes to reach that point.

The polar coordinate system is a different way to represent points on the plane by a pair $(r, \theta)$:




Here we have a fixed axis ray emanating from origin O, and we reach a point $(r, \theta)$ by making an angle of $\theta$ radians and going out a distance of $r$.

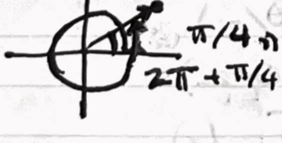
E.g. The point $(x,y) = (1,1)$ in Cartesian coord's is the same as $(r, \theta) = (\sqrt{2}, \frac{\pi}{4})$ in polar coord's:



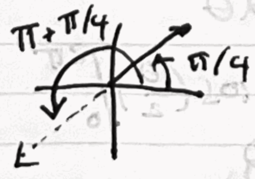

length of hypotenuse $= \sqrt{1^2+1^2} = \sqrt{2}$

Notice: There are multiple ways to represent any point in polar coord's because we can add $2\pi$ to $\theta$:

$(r, \theta) = (\sqrt{2}, \pi/4)$ same as $(r, \theta) = (\sqrt{2}, 2\pi + \pi/4)$



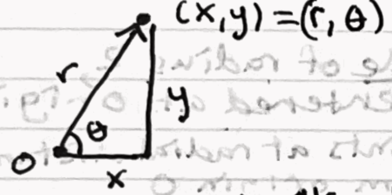

Also ... can add $\pi$ to $\theta$ and replace $r$ by $-r$:




$(r, \theta) = (\sqrt{2}, \pi/4)$ same as $(r, \theta) = (-\sqrt{2}, \pi + \pi/4)$

Negative value of $r$ means go backwards that distance along ray.

Question: How to convert between Cartesian & polar coord's?
Let's draw a right triangle to help us:




⇐ From this picture we see that $\boxed{x = r\cos\theta \text{ and } y = r\sin\theta}$ which gives $(x,y)$ in terms of $(r,\theta)$

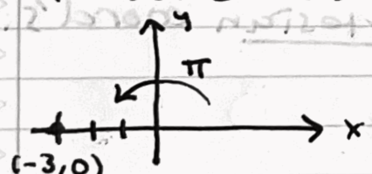
We also have that:

$$\boxed{r^2 = x^2 + y^2 \text{ and } \tan\theta = \frac{y}{x}}$$

which gives us $(r,\theta)$ in terms of $(x,y)$:
specifically, $r = \pm\sqrt{x^2+y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$.

---

E.g. Find the polar coordinates of $(x,y) = (-3,0)$.
To solve this problem, it's easiest to just draw the point.




we see this point is at angle $\theta = \pi$ and radius $r = 3$.

Check: $3^2 = r^2 = x^2 + y^2 = (-3)^2 + (0)^2$
and $0 = \tan(\theta) = \frac{y}{x} = \frac{0}{-3}$.

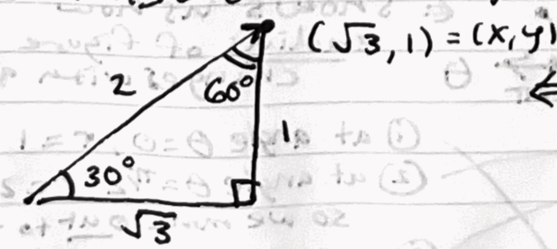
Could have also chosen $(r,\theta) = (-3, 0)$ here...

---

E.g. Find the Cartesian coordinates of $(r,\theta) = (2, \frac{\pi}{6})$.

Here we have $x = r\cos\theta = 2\cos(\pi/6) = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$
and $y = r\sin\theta = 2\sin(\pi/6) = 2\cdot\frac{1}{2} = 1$

Can also draw the right triangle to check:



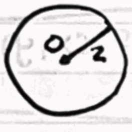

⇐ recall that $\theta = \pi/6$ radians $= 30°$ corresponds to a special "30-60-90" triangle.

3/18

Polar equations and curves:

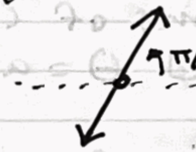
Just like we draw curves $f(x,y)=0$ in Cartesian coord's, we can draw curves $f(r,\theta)=0$ in Polar coord's.

E.g. The equation $r=2$ gives circle of radius 2, centered at origin:




⇐ circle = all points at radial distance 2 from origin O.

E.g. The equation $\theta = \pi/3$ gives line at angle $\pi/3$ thru origin:




⇐ line thru origin = all points at given angle

E.g. what about equation $r = 2\cos\theta$?

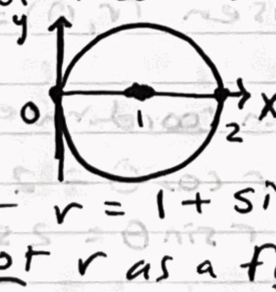
Here it's easiest to switch to Cartesian coord's:

multiplying by r gives $r^2 = 2r\cos\theta$

$\Leftrightarrow x^2+y^2 = 2x$

$\Leftrightarrow (x-1)^2+y^2 = 1$

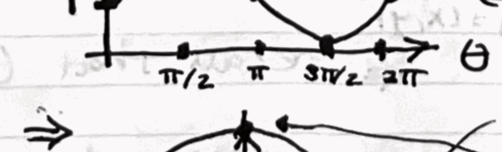
which is a circle of radius 1 centered at $(x,y)=(1,0)$:




⇐ $(x-1)^2+y^2=1$ a.k.a. $r = 2\cos\theta$

E.g. What about $r = 1+\sin(\theta)$?

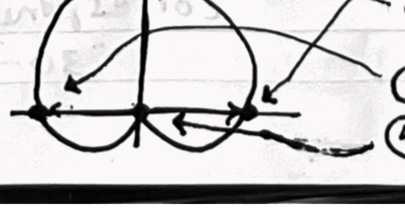
First let's plot r as a function of $\theta$ (in Cartesian coord's):




⇐ shows us how radius of figure changes with angle

"cardioid" ⇒

this "heart-shaped" curve is polar curve $r = 1+\sin(\theta)$



① at angle $\theta=0$, $r=1$

② at angle $\theta=\pi/2$, $r=2$ so we move out to this point

③ at $\theta=\pi$, back to $r=1$

④ at $\theta=\frac{3\pi}{2}$, radius shrinks to $r=0$

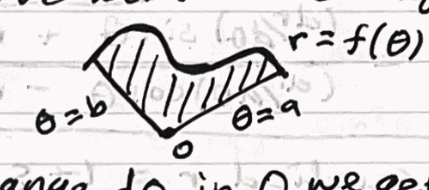
3/20

# Calculus in Polar coordinates §10.4

We can do all types of calculus stuff in polar coord's too...

Areas: How to compute area "inside" polar curve $r = f(\theta)$? where $a \leq \theta \leq b$

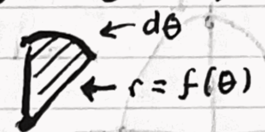
The polar curve looks something like this:




For a small change $d\theta$ in $\theta$ we get roughly a "pie slice":

$\text{area} = \pi r^2 \cdot \frac{d\theta}{2\pi} \Rightarrow$

$= \frac{1}{2}(f(\theta))^2\, d\theta$

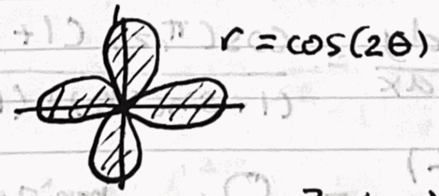



As usual, breaking up area into many small pie slices and summing up area gives an integral in limit:

$$\text{area inside polar curve} = \boxed{\int_a^b \frac{1}{2}(f(\theta))^2\, d\theta}$$

E.g. Let's look at the curve $r = \cos(2\theta)$ for $0 \leq \theta \leq 2\pi$:

"four-leaf clover" shape $\Rightarrow$




What is area inside this curve? Using formula...

$$\text{Area} = \int_0^{2\pi} \frac{1}{2}(f(\theta))^2 d\theta = \int_0^{2\pi} \frac{1}{2}\cos^2 2\theta\, d\theta$$

We've seen before (using int. by parts) that $\int \cos^2 x\, dx = \frac{1}{2}(x + \sin(x)\cos(x))$

So w/ a simple u-sub $\int \frac{1}{2}\cos^2 2\theta\, d\theta = \frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\cos(2\theta)$

Thus, area $= \int_0^{2\pi} \frac{1}{2}\cos^2 2\theta\, d\theta = \left[\frac{1}{4}\theta + \frac{1}{8}\sin(2\theta)\cos(2\theta)\right]_0^{2\pi}$

$= \left(\left(\frac{1}{4}\cdot 2\pi + \frac{1}{8}\overset{0}{\sin(4\pi)}\cos(4\pi)\right) - \left(\frac{1}{4}\cdot 0 + \frac{1}{8}\overset{0}{\sin(0)}\cos(0)\right)\right) = \boxed{\frac{\pi}{2}}$

**Arc lengths**: How to compute length of polar curve $r=f(\theta)$?

As before, from $x=r\cos\theta$ and $y=r\sin\theta$ we get

$$\frac{dx}{d\theta}=\frac{dr}{d\theta}\cos\theta-r\sin\theta \quad\text{and}\quad \frac{dy}{d\theta}=\frac{dr}{d\theta}\sin\theta+r\cos\theta$$

so that

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left(\frac{dr}{d\theta}\right)^2\cos^2\theta-2r\frac{dr}{d\theta}\cos\theta\sin\theta+r^2\sin^2\theta$$
$$+\left(\frac{dr}{d\theta}\right)^2\sin^2\theta+2r\frac{dr}{d\theta}\sin\theta\cos\theta+r^2\cos^2\theta$$
$$=\left(\frac{dr}{d\theta}\right)^2+r^2 \quad (\text{using } \sin^2\theta+\cos^2\theta=1)$$

If we think of $(x,y)$ as parametrized by $\theta$, then

$$\text{length of curve}=\int_a^b\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

which in terms of $r$ and $\theta$ is then

$$\text{length}=\boxed{\int_a^b\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}$$

E.g. For a circle $r=m$ centered at origin,

this formula gives us $\text{length}=\int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int_0^{2\pi}\sqrt{m^2+0^2}\,d\theta$

$$=\int_0^{2\pi}m\,d\theta=2\pi m,$$

which is correct circumfrence!

E.g. We saw before that $r=2\cos\theta$, $0\le\theta\le\pi$ gives a circle of radius 1 centered at $(x,y)=(1,0)$

Here $dr/d\theta=-2\sin\theta$, so the formula gives...

$$\text{arc length}=\int_0^{\pi}\sqrt{(2\cos\theta)^2+(-2\sin\theta)^2}\,d\theta=\int_0^{\pi}2\,d\theta=2\pi.\ ✓$$

//

3/22

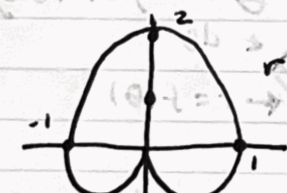
**Tangents**: How to find slope of tangent to polar curve $r = f(\theta)$?

Recall $x = r\cos\theta$ and $y = r\sin\theta$ in Cartesian coord's.
So using the product rule we get:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \quad \text{and} \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta}}$$

E.g. Consider the cardioid $r = 1 + \sin\theta$:



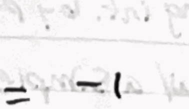

$r = 1 + \sin\theta$
$\Rightarrow dr/d\theta = \cos\theta$

Here
$$\frac{dy}{dx} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

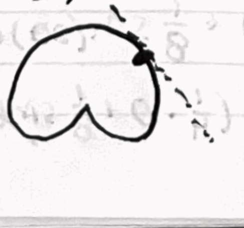
So at $\theta = \frac{\pi}{2}$ get
$$\frac{dy}{dx} = \frac{\cos(\pi/2)\,(1 + 2\sin(\pi/2))}{(1+\sin(\pi/2))(1 - 2\sin(\pi/2))}$$

$$= \frac{0(1+2)}{(1+1)(1-2)} = 0 \rightarrow \text{horizontal tangent at } \theta = \pi/2$$




And at $\theta = \pi/3$ get
$$\frac{dy}{dx} = \frac{\cos(\pi/3)\,(1 + 2\sin(\pi/3))}{(1+\sin(\pi/3))(1 - 2\sin(\pi/3))}$$

$$= \frac{(1/2)(1 + 2\frac{\sqrt{3}}{2})}{(1+\frac{\sqrt{3}}{2})(1 - 2\sqrt{3}/2)} = \frac{1+\sqrt{3}}{(2+\sqrt{3})(1-\sqrt{3})} = \frac{1+\sqrt{3}}{-1-\sqrt{3}} = -1$$



← tangent slope $= -1$ at $\theta = \pi/3$