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Sequences § 11.1

We now start a new chapter, Ch. 11, on sequences, series, and power series. This is the final topic of the semester.

Def'n An (infinite) sequence is an infinite list $a_1, a_2, a_3, \ldots, a_n, \ldots$ of real numbers. We also use $\{a_n\}$ and $\{a_n\}_{n=1}^{\infty}$ to denote this sequence.

E.g. We can let $a_n = \frac{1}{2^n}$ for $n \geq 1$, which gives the sequence $1/2, 1/4, 1/8, 1/16, 1/32, \ldots$

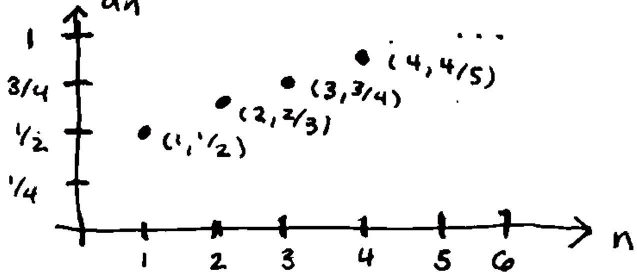
E.g. $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$

Can also write $\left\{\frac{n}{n+1}\right\}_{n=2}^{\infty} = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ to start at $n=2$, or also $\left\{\frac{n+1}{n+2}\right\}_{n=1}^{\infty} = \left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$.

E.g. Not all sequences have simple formulas for the n^{th} term. For example, with $a_n = n^{\text{th}}$ digit of π after the decimal point, we have $\{a_n\} = \{1, 4, 1, 5, 9, 2, 6, 5, \ldots\}$ but there is no easy way to get the n^{th} term here...

Def'n The graph of sequence $\{a_n\}_{n=1}^{\infty}$ is the collection of points $(1, a_1), (2, a_2), (3, a_3), \ldots$ in the plane.

E.g. For the sequence $a_n = \frac{n}{n+1}$, its graph is

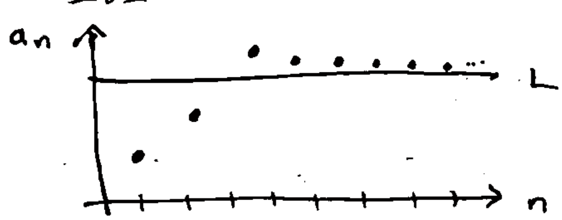


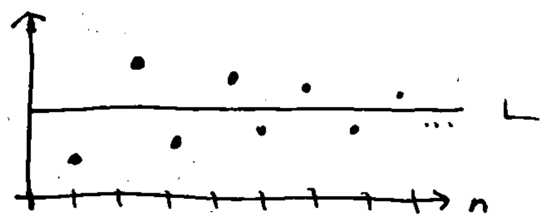
The graph of a sequence is like the graph of a function, but we get discrete points instead of a continuous curve. Notice how for this graph, points approach line $y=1$ ...

Def'n We say the limit of sequence $\{a_n\}$ is L, written "$\lim_{n\to\infty} a_n = L$" or "$a_n \to L$ as $n \to \infty$" if, intuitively, we can make the terms a_n as close to L as we'd like by taking n sufficiently large.
(Precise definition uses ε, like limits in Calc I...)
If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges.
Otherwise, we say the sequence diverges.

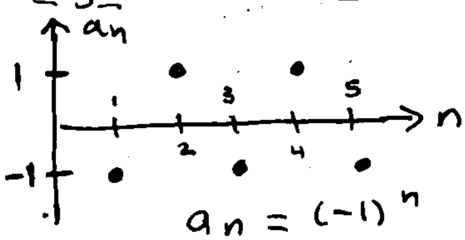
E.g. The sequence $a_n = \frac{n}{n+1}$ has $\lim_{n\to\infty} a_n = 1$ (we'll prove this later...)

E.g. Some other convergent sequences look like:

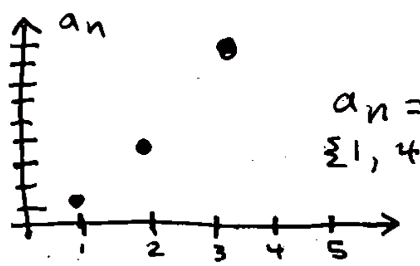




E.g. Some divergent sequences are:



$a_n = (-1)^n$
$\{-1, 1, -1, 1, \dots\}$



$a_n = n^2$
$\{1, 4, 9, 16, \dots\}$

Notice how this 2^{nd} example $a_n = n^2$ "goes off to ∞."

Def'n The notation "$\lim_{n\to\infty} a_n = \infty$" means that for every M there is an N such that $a_n > M$ for all $n > N$.
We define "$\lim_{n\to\infty} a_n = -\infty$" similarly.

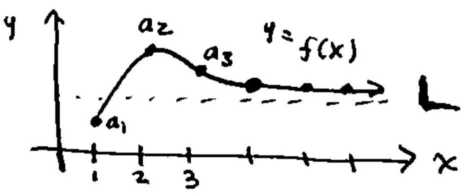
E.g. $\lim_{n\to\infty} n^2 = \infty$ and $\lim_{n\to\infty} -n = -\infty$.

Having an infinite limit is one way a sequence can diverge.

Limits of sequences are very similar to limits of functions:

Theorem If $f(x)$ is a function with $f(n) = a_n$ for all positive integers n, then if $\lim_{x\to\infty} f(x) = L$ also $\lim_{n\to\infty} a_n = L$.

Picture:



E.g. How to find $\lim_{n\to\infty} \frac{\ln(n)}{n}$? Instead, let $f(x) = \frac{\ln(x)}{x}$, then $\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1}$ (by L'Hôpital's Rule)

$$= \lim_{x\to\infty} 1/x = 0$$

So we also have that $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$.

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All the basic rules for limits of functions apply to sequences:

Theorem (Limit Laws for Sequences)

For convergent sequences $\{a_n\}$ and $\{b_n\}$, we have:

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.
- $\lim_{n\to\infty} (c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n$ for any constant $c \in \mathbb{R}$.
- $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = (\lim_{n\to\infty} a_n) / (\lim_{n\to\infty} b_n)$ if $\lim_{n\to\infty} b_n \neq 0$.

E.g. To compute $\lim_{n\to\infty} \frac{n}{n+1}$, we can use these rules:

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} = \frac{1}{1+0} = 1$$

multiply top and bottom by $1/n$

as claimed! ✓

Another very useful lemma for computing limits of sequences:

Lemma If $\lim_{n\to\infty} a_n = L$ and $f(x)$ is continuous at $x = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$.

E.g. Q: What is $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right)$?

A: Notice $\lim_{n\to\infty} \frac{\pi}{n} = 0$ and cos is continuous at 0, so that $\lim_{n\to\infty} \cos(\pi/n) = \cos(0) = 1$.

Another useful lemma for limits of sequences with signs:

Lemma If $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$.

E.g. How to compute $\lim_{n\to\infty} \frac{(-1)^n}{n}$? Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we also have that $\lim_{n\to\infty} (-1)^n/n = 0$.

Compare this to $a_n = (-1)^n$, which diverges!

One of the most important kind of sequences are the sequences $a_n = r^n$ for some fixed number $r \in \mathbb{R}$. When does this sequence converge?

We have seen in Calc I that for $0 < r < 1$,

$$\lim_{x\to\infty} (r^x) = 0 \quad (\text{think: } \lim_{x\to\infty} (\tfrac{1}{2})^x = 0)$$

So $\lim_{n\to\infty} r^n = 0$ for $0 < r < 1$ too.

By the absolute value lemma, $\lim_{n\to\infty} r^n = 0$ when $-1 < r < 0$ as well.

Also, clearly $\lim_{n\to\infty} 1^n = \lim_{n\to\infty} 1 = 1$. But other r diverge!

Theorem

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{for all other } r. \end{cases}$$

Monotone and bounded sequences §11.1

Def'n The sequence $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$, and decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called monotone if it is either increasing or decreasing.

E.g. The sequence $a_n = n$ is increasing (hence monotone). The sequence $a_n = (-1)^n$ is neither increasing nor decreasing.

Def'n $\{a_n\}$ is bounded above if there is some M such that $a_n < M$ for all $n \geq 1$, it is bounded below if there is M such that $a_n > M$ for all $n \geq 1$, and it is bounded if it is both bounded above and below.

E.g. $a_n = (-1)^n$ is bounded (above by 2 and below by -2), but $a_n = n$ is unbounded since it goes off to ∞.

Clearly a sequence which is unbounded (like $a_n = n$) cannot be convergent. Some bounded sequences, (like $a_n = (-1)^n$), are also divergent. But if our sequence is both bounded and monotone, then it must converge:

Thm (Monotone Sequence Theorem) Every bounded, monotone (either increasing or decreasing) sequence converges.

Picture proof:

$\Leftarrow$ an increasing sequence bounded by M will converge to an L with $L \leq M$.

E.g. $a_n = \frac{1}{n}$ is bounded and monotone (decreasing) so it converges, as we were already aware.

Exercise Use the Monotone Convergence Theorem to explain why $a_n = \frac{n}{n+1}$ converges (which we also knew...)

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<u>Series</u> §11.2

A <u>series</u> is basically an "infinite sum."
If we have an (infinite) sequence $\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$
the corresponding <u>series</u> is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

An infinite sum like this does not always make sense:

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + 5 + \cdots = \text{“}\infty\text{”}$$

But sometimes we can sum ∞-many terms & get a finite number:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = ???$$

Well, $\frac{1}{2} = 0.5$, $\frac{1}{2} + \frac{1}{4} = 0.75$, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$,
and it seems that if we add up more and more terms,
we don't go off to ∞, but instead get closer and closer to $\underline{\underline{1}}$.

<u>Def'n</u> For <u>series</u> $\sum_{n=1}^{\infty} a_n$, the associated <u>partial sums</u>
are $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$ for $n \geq 1$.
If $\lim_{n\to\infty} S_n = L$ then we write $\sum_{n=1}^{\infty} a_n = L$ and we
say the series <u>converges</u>. Otherwise, it <u>diverges</u>.

<u>key idea</u>: $\boxed{\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n)}$

<u>E.g.</u>: Let $a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. What is $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$?

Well, $S_n = \underbrace{\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right)}_{a_1} + \underbrace{\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right)}_{a_2} + \cancel{\cdots} + \underbrace{\left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right)}_{a_{n-1}} + \underbrace{\left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)}_{a_n}$

$= 1 - \frac{1}{n+1}$, so that $\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = \underline{\underline{1}}$.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \underline{\underline{1}}$.

One of the most important kind of series are the <u>geometric series</u>:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots, \quad \text{for real numbers } a \text{ and } r \neq 0.$$

Notice that $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$

and $r \cdot S_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$

$\Rightarrow \quad (1-r) \cdot S_n = a \qquad\qquad - ar^n$

$\Rightarrow \quad S_n = \dfrac{a - ar^n}{(1-r)}$

Since $\lim_{n\to\infty} r^n = 0$ for $|r|<1$, we have:

$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \frac{a}{1-r} \quad \text{for } |r|<1.$$

<u>important formula to remember</u>: value of geo. series when $|r|<1$.

<u>E.g.</u>: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is geo. series with $a = 1/2$ and $r = 1/2$. So $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1$.

This is what we expected above!

For $|r| \geq 1$, geo. series $\sum_{n=1}^{\infty} ar^{n-1}$ <u>diverges</u>.

Consider in particular the case $a = r = 1$.

Then $\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots$, so the partial sums are $S_n = 1 + 1 + \cdots + 1 = n$, and $\lim_{n\to\infty} S_n = \infty$.

In general, in order to converge, the terms in a series must <u>approach zero</u>:

<u>Theorem</u> (Divergence Test) If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. So if $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ <u>diverges</u>.

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WARNING: The divergence test says that if terms do not go to zero, the series diverges.
But converse does not hold: the terms a_n can go to 0, while the series $\sum_{n=1}^{\infty} a_n$ still diverges.

The most important (counter)example is the harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

Of course, $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ still diverges.
How to see this? Ignore the 1 at the start, and consider

$$\underbrace{\frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 2\cdot\frac{1}{4} = \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 4\cdot\frac{1}{8} = \frac{1}{2}} + \cdots \quad \cdots$$

The trick, as shown above, is to break the series into chunks consisting of 1, 2, 4, 8, ... terms.
If we add up the terms in each chunk, we get a sum bigger than $\frac{1}{2}$. So overall sum is $\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$
But a sum of ∞-many $\frac{1}{2}$'s must diverge!
So the harmonic series, which is bigger than that, diverges too.

Theorem (Laws for series)
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge.
Then • $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
and • $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$ for any $c \in \mathbb{R}$.

WARNING: $\sum_{n=1}^{\infty} a_n \cdot b_n \neq \left(\sum a_n\right) \cdot \left(\sum_{n=1} b_n\right)$.

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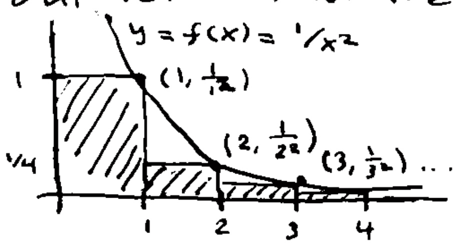
Integral test for convergence §11.3

We saw a couple series whose convergence we could establish because we had a simple formula for the partial sums. That's not possible for most series. We need tools to study convergence.

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. There's no simple formula for its partial sums. But let's draw the following picture:



← plot the sequence $a_n = \frac{1}{n^2}$ and use this to make rectangles of width $= 1$ and height $= a_n$

Notice that the area of the n^{th} rectangle $= a_n \times 1 = a_n$,
So the sum of areas $= a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$.
Also notice that we plotted the curve $y = f(x) = \frac{1}{x^2}$.
The area under $y = f(x)$ from $x = 1$ to ∞ is visibly less than $a_2 + a_3 + a_4 + \dots = \left(\sum_{n=1}^{\infty} a_n\right) - a_1$.
But we can compute area under $y = f(x)$ as an improper integral:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t 1/x^2\,dx = \lim_{t\to\infty} \left[-1/x\right]_1^t = \lim_{t\to\infty} \left(1 - \frac{1}{t}\right) = \underline{\underline{1}}.$$

Thus, $\sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} 1/x^2\,dx = 1 + 1 = 2$, so in particular this series converges: it has a finite value.
(Since all the terms are positive, if it diverged it would go off to ∞. Being bounded means it converges.)

This way of comparing a series to an associated integral is called the integral test for convergence.
It can be used to establish divergence as well:

Theorem (Integral Test for Convergence / Divergence)
Let $f(x)$ be a continuous, positive, (eventually) decreasing function on $[1,\infty)$, and let $a_n = f(n)$ for all $n \geq 1$.
1) If $\int_1^\infty f(x)\,dx$ converges, then $\sum_{n=1}^\infty a_n$ converges.
2) If $\int_1^\infty f(x)\,dx$ diverges, then $\sum_{n=1}^\infty a_n$ diverges.

E.g. We saw before that harmonic series $\sum_{n=1}^\infty 1/n$ diverges.
We can also prove this using the integral test:
$$\int_1^\infty 1/x \, dx = \lim_{t\to\infty} \int_1^t 1/x \, dx = \lim_{t\to\infty} \left[\ln(x)\right]_1^t = \lim_{t\to\infty} \ln(t) = \underline{\underline{\infty}}.$$

Comparing $\sum_{n=1}^\infty \frac{1}{n}$ and $\sum_{n=1}^\infty \frac{1}{n^2}$, a natural question is:
for which values of p does the series $\sum_{n=1}^\infty \frac{1}{n^p}$ converge?
(The book calls these series "p-series".)

Theorem The series $\sum_{n=1}^\infty \frac{1}{n^p}$:
- diverges for $p \leq 1$
- converges for $p > 1$.

Pf: First note that if $p \leq 0$ then $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$,
So the series diverges by the Test for Divergence.
So suppose $0 < p < 1$. Then $\int \frac{1}{x^p}\,dx = \frac{1}{1-p} x^{1-p}$
So that $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \left[\frac{1}{1-p} x^{1-p}\right]_1^t = \infty$,
So the series diverges by the Integral Test.
We have already seen that $\sum_{n=1}^\infty 1/n$ diverges.
So finally assume $p > 1$. Then $\int \frac{1}{x^p}\,dx = \frac{-1}{(p-1)x^{p-1}}$
So that $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \left[\frac{-1}{(p-1)x^{p-1}}\right]_1^t = \frac{1}{p-1}$,
So the series converges by the Integral Test. ▨

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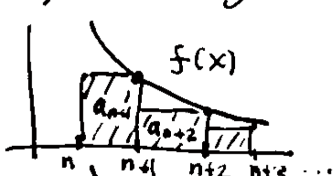
Estimating Remainders with Integrals §11.3

Integrals are useful for establishing convergence of series, but don't tell us the exact value of the series. Still, they can be used to estimate the value of the series.

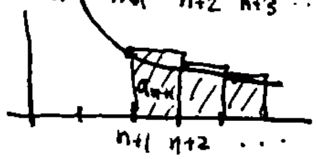
As above, let $f(x)$ be a continuous, positive, decreasing fn. on $[1,\infty)$ and let $a_n = f(n)$ for all $n \geq 1$. We want to estimate value of series $s = \sum_{n=1}^{\infty} a_n$. A simple estimate for any series is a partial sum $s_n = a_1 + a_2 + \dots + a_n$, for some finite value of n. How good of an estimate is s_n for the true value of the series s? Define the remainder to be $R_n = s - s_n$.

E.g. For $s = \sum_{n=1}^{\infty} \frac{1}{2^n}$, have $s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, and know $s = 1$, so $R_2 = \frac{1}{4}$.

By looking at the two pictures below:



⇐ overestimate
$$R_n = a_{n+1} + a_{n+2} + \dots \leq \int_n^{\infty} f(x)\,dx$$



⇐ underestimate
$$R_n = a_{n+1} + a_{n+2} + \dots \geq \int_{n+1}^{\infty} f(x)\,dx$$

Theorem We have $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$

E.g. For $s = \sum_{n=1}^{\infty} \frac{1}{n^2}$, $s_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.42$, and by thm:

$$\int_5^{\infty} \frac{1}{x^2}\,dx \leq R_4 \leq \int_4^{\infty} \frac{1}{x^2}\,dx$$
$$1/5 \leq R_4 \leq 1/4$$
$$0.2 \leq s - 1.42 \leq 0.25$$
$$1.62 \leq s \leq 1.67 \leftarrow \text{good estimate of } \sum_{n=1}^{\infty} \frac{1}{n^2}$$

(In fact, $s = \frac{\pi^2}{6} \approx 1.64\ldots$, but that result is beyond this class...)

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Comparison Tests for Series §11.4

We know the geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges ($|r|<1$).

The series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ seems very similar, but how can we show it converges or diverges? In fact, we can compare the two series:

Theorem (Direct Comparison Test for series)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series whose terms are all positive. Then:

1) If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \le b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ converges too.

2) If $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \ge b_n$ for all n then $\sum_{n=1}^{\infty} a_n$ diverges too.

Note: positive terms here is very important!

E.g: Notice that $\frac{1}{2^n+1} \le \frac{1}{2^n}$ for all $n \ge 1$,
(dividing 1 by a bigger number gives something smaller)
So therefore $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ also converges.

E.g: Easy to show that if $\sum_{n=1}^{\infty} a_n$ diverges/converges, then $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$ also diverges/converges for any nonzero scalar $c \in \mathbb{R} \setminus \{0\}$.

So $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$ also diverges, like harmonic series.

And then notice $\frac{1}{2n-1} \ge \frac{1}{2n}$ for all $n \ge 1$,
So therefore $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ also diverges by direct comparison.

The series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ also seems very similar to $\sum_{n=1}^{\infty} \frac{1}{2^n}$,
so we expect that it would also converge.
Unfortunately, $\frac{1}{2^n-1} > \frac{1}{2^n}$ for all $n \geq 1$, wrong direction of inequality to show convergence by direct comparison.

Instead we can use the following:

Theorem (Limit Comparison Test for Series)
Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with positive terms.
Suppose $c = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists and $c \neq 0$ and $c \neq \pm\infty$.
Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

E.g. Notice $\lim_{n\to\infty} \frac{1/2^n}{1/(2^n-1)} = \lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} \frac{1-\frac{1}{2^n}}{1} = 1$,
so since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, by Limit Comparison $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges too.

E.g. Consider a series like $\sum_{n=1}^{\infty} \frac{8n}{5n^2+n-1}$.
How to decide convergence/divergence? Compare to $\sum_{n=1}^{\infty} \frac{1}{n}$:
$\lim_{n\to\infty} \left(\frac{8n}{5n^2+n-1}\right) / \left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{8n^2}{5n^2+n-1} = \frac{8}{5}$, so
by limit comparison, $\sum_{n=1}^{\infty} \frac{8n}{5n^2+n-1}$ also diverges.

Exercise: Show $\sum_{n=1}^{\infty} \frac{8n}{5n^3+n-1}$ converges
Hint: use Limit Comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Key takeaway: For series whose terms are rational functions, check biggest power of n on top vs. on bottom.

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Alternating Series §11.5

The convergence tests we've seen (integral test, comparison) are for series with positive terms only.

Things become more complicated when there are signs.

The most important kind of series with signs are the alternating series, where terms switch positive to negative to positive, to negative, etc.

like $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$

or $\sum_{n=1}^{\infty}(-1)^{n}\frac{3n}{4n-1} = \frac{-3}{3}+\frac{6}{7}-\frac{9}{11}+\frac{12}{15}-\cdots$

As we can see, an alternating series has the form:

$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$ where b_n is a sequence of positive numbers

(which form it has depends on if it starts positive or negative)

Theorem (Alternating Series Test)

For an alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1-b_2+b_3-b_4+\cdots$ where $b_n>0$ are all positive, if we have that:

- $b_{n+1}\le b_n$ for all $n\ge 1$ (terms are getting smaller)
- $\lim_{n\to\infty} b_n = 0$ (terms go to zero),

then the series converges.

E.g. The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ satisfies these conditions: • $\frac{1}{n+1}<\frac{1}{n}$

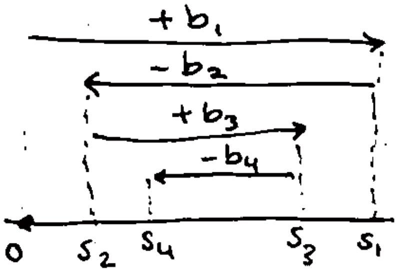
• and $\lim_{n\to\infty}\frac{1}{n}=0$

So $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$ converges, unlike usual harmonic series.

Idea: terms cancel each other, so sum more likely to converge!

Skipped!

Picture proof of Alternating Series Test:



We start with 0. We add $+b_1$ to get s_1. Then we subtract $-b_2$ to get s_2. Etc. But we never go back further than where we just were, since $b_{n+1} \le b_n$. So we get "trapped" in a smaller and smaller space, as the $b_n \to 0$ when $n \to \infty$. Thus, the series must converge. ∎

In fact, we can use this argument to estimate the series:

Thm Let $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ be an alternating series, satisfying conditions: • $b_{n+1} \le b_n$ for all n & • $\lim_{n\to\infty} b_n = 0$.
Let $s_n = b_1 - b_2 + b_3 - \cdots \pm b_n$ be the nth partial sum and $R_n = s - s_n$ be the remainder (error) of this partial sum.
Then $|R_n| (= |s - s_n|) \le b_{n+1}$.
"Error is bounded by next term."

E.g. Let's compute $s = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ accurately to within 0.1.
We compute $s_9 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} = \frac{1879}{2520} \approx 0.745\ldots$
and by thm, $|R_n| \le \frac{1}{10}$ (next term), so $s \approx 0.745\ldots \pm 0.1$.

E.g. Decide if the alternating series
$$\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1}$$ converges or diverges.

Here: $\lim_{n\to\infty} \frac{3n}{4n-1} = \frac{3}{4} \ne 0$, so we cannot use the alternating series test to establish convergence.
Actually, $\lim_{n\to\infty} (-1)^n \frac{3n}{4n-1}$ does not exist, so by the divergence test (terms don't go to 0) series diverges!

Absolute convergence vs. Conditional convergence

Def'n A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ (series of absolute values) converges.

Thm If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Def'n Series is called conditionally convergent if it is convergent but not absolutely convergent.

E.g. The alternating harmonic series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ is conditionally convergent, since it converges, but $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) diverges.

The Ratio Test § 11.6

For a geometric series $\sum_{n=1}^{\infty} a r^{n-1}$, convergence/divergence is determined by ratio $|a_{n+1}/a_n| = |r|$ of terms.

In fact, this is important for any series:

Theorem (Ratio Test for Absolute Convergence)

For series $\sum_{n=1}^{\infty} a_n$, let $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ (limiting ratio of successive terms).

If $L < 1$, then the series converges absolutely, (and hence converges),

If $L > 1$ (including $L = \infty$), then the series diverges.

If $L = 1$, the test is inconclusive (could go either way).

Pf idea: We compare the series to geometric series $\sum_{n=1}^{\infty} L^{n-1}$, which converges if $L < 1$ and diverges if $L > 1$. ∎

E.g. Does the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ converge absolutely?

Here $|a_n| = \frac{n^3}{3^n}$, so $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^3/3^{n+1}}{n^3/3^n}$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{3^n} \cdot \frac{3^n}{n^3} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1+\frac{1}{n}\right)^3 = \frac{1}{3}(1+0)^3 = \frac{1}{3}.$$

Since $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{1}{3} < 1$, this series converges absolutely.

The ratio test is useful when the series has terms involving $2^n, 3^n, e^n$, etc. that are exponential in n. These factors are "more important" than polynomials.

E.g. Let's try applying the ratio test to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Here $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = \underline{1}$.

So the ratio test is inconclusive: it fails to detect whether $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges (we know it converges).

In fact, ratio test fails for any p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

This makes sense, since most of the examples of conditionally convergent series we know are related to p-series, and the ratio test cannot detect conditional convergence.

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Power series § 11.8

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Here the c_n are a sequence of numbers we call coefficients, while "x" is a variable, which we can specialize to any number.

For example, if $c_n = 1$ for all $n \geq 0$, then we get

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

the geometric series with ratio x, which converges $\Leftrightarrow |x| < 1$.

We can think of the power series as defining a function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

which is defined for x such that the series converges.

for example, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$.

More generally, for a number a, we can consider a power series centered at a, which is a series of form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

E.g. Find the values of x for which the power series

$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converges. Idea: use ratio test.

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-3|^{n+1}}{n+1} \cdot \frac{n}{|x-3|^n} = \lim_{n\to\infty} |x-3| \left(\frac{n}{n+1}\right) = |x-3|.$$

So when $|x-3| < 1$, series converges & when $|x-3| > 1$, series diverges.

Notice $|x-3| < 1 \Leftrightarrow 2 < x < 4$. For $x = 2$ and $x = 4$, ratio test inconclusive.

But $x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ alt. harmonic series $\Rightarrow$ converges and $x = 4 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series $\Rightarrow$ diverges.

In summary, this series converges exactly for $2 \leq x < 4$.

Thm For power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, only three things can happen:

i) The series converges only when $x=a$. ("$R=0$")

ii) The series converges for all x. ("$R=\infty$")

iii) There is a positive number R such that the series converges when $|x-a|<R$ and diverges when $|x-a|>R$.

Pf idea: Ratio test, like last example. ☒

The number R in the above thm called radius of convergence (where we declare $R=0$ in case i) and $R=\infty$ in case ii)).

The interval $a-R \underset{?}{<} x \underset{?}{<} a+R$ is called the interval of convergence of the series.

WARNING: Whether series converges at endpoints $a-R, a+R$ is tricky, usually have to use something beyond ratio test.

E.g.: For n a positive integer, the number n factorial is $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$ (and $0! = 1$).

Consider power series centered at 0 w/ coeff's $c_n = \frac{1}{n!}$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

Let's find the radius of convergence of this series.

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{(n+1)}$$

For any fixed x, $(n+1)$ is eventually much bigger than $|x|$,

so $L = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$ for every x.

Thus, Ratio Test says that $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges for all $x \in \mathbb{R}$, i.e., radius of convergence is $R=\infty$.

Exercise: Show radius of convergence of $\sum_{n=0}^{\infty} n!\, x^n$ is $\underline{0}$. //

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Representing functions as power series §11.9

We have seen that $\sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots = \frac{1}{1-x}$ for $|x|<1$.

So we can represent the function $f(x) = \frac{1}{1-x}$ as a power series $f(x) = \sum_{n=0}^{\infty} x^n$, at least for $|x|<1$.

Another way to think about this: we have the partial sums $S_n(x) = 1+x+x^2+\cdots+x^n$, which are polynomials in x.

And $f(x) = \sum_{n=0}^{\infty} x^n$ means $f(x) = \lim_{n\to\infty} S_n(x)$ for $|x|<1$.

We can represent many other functions (especially rational functions) as power series, via algebraic manipulations:

E.g.: How to write $f(x) = \frac{1}{1+x^2}$ as a power series?

write $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

$$= 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

important substitution technique!

This geometric series converges for $|(-x^2)|<1$, i.e., $|x|<1$.

E.g.: How to find power series representation of $f(x) = \frac{1}{x+2}$?

Write $\frac{1}{2+x} = \frac{1}{2(1+\frac{x}{2})} = \frac{1}{2}\cdot\frac{1}{1+\frac{x}{2}} = \frac{1}{2}\frac{1}{1-(\frac{-x}{2})}$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This geo. series converges for $|\frac{-x}{2}|<1$, i.e., $|x|<2$ meaning for $x\in(-2,2)$.

E.g.: What about $f(x) = \frac{x^3}{x+2}$? Here we write:

$$\frac{x^3}{x+2} = x^3\cdot\frac{1}{x+2} = x^3\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{n+3}$$

$$= \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \cdots$$

As in previous example, the interval of convergence is $(-2,2)$. //

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Differentiating and Integrating Power Series §11.9

Thm If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series at $x=a$ with nonzero radius of convergence $R>0$, then

(i) $f'(x) = \sum_{n=0}^{\infty} n \cdot c_n (x-a)^{n-1}$ is the derivative,

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$ is the integral (where C is any constant),

and these power series also have radius of convergence $R>0$.

Note: This is saying we can differentiate/integrate power series "as though they were polynomials":

$$\frac{d}{dx}(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

$$\int c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \, dx = C + c_0 x + \frac{c_1}{2} x^2 + \frac{c_2}{3} x^3 + \dots$$

E.g. We know that $\frac{1}{1-x} = 1 + x + x^2 + x^2 + \dots = \sum_{n=0}^{\infty} x^n$.

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left((1-x)^{-1}\right) = -(1-x)^{-2}\cdot -1 = \frac{1}{(1-x)^2}$$

so the rule for differentiating power series says

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \dots = \sum_{n=0}^{\infty} (n+1) x^n.$$

E.g. How to find power series representation of $\ln(1+x)$?

Notice that $\int \ln(1+x)\,dx = \frac{1}{1+x}$ and we know

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n,$$

so by the rule for integrating power series we get

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}\right) + C$$

$$= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

At $x=0$, have $\ln(1+0) = 0$, so integration constant is $C=0$

$$\Rightarrow \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

In both above examples, radius of convergence is $R = 1$. //

Taylor Series §11.10

Let $f(x)$ be infinitely-differentiable in an interval containing $x=a$. Use $f^{(n)}(x)$ to mean the n^{th} derivative of $f(x)$:

$$f^{(0)}(x) = f(x),\ f^{(1)}(x) = f'(x),\ f^{(2)}(x) = f''(x),\ \text{etc.}$$

<u>Def'n</u> The <u>Taylor series</u> of $f(x)$ at $x=a$ is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Most important case is when $a=0$, and then is called <u>Taylor-Maclaurin</u> (or just <u>Maclaurin</u>) <u>series</u>:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Why do we define Taylor series like this? Look at what happens when we take n derivatives:

$$\frac{d^n}{dx^n}\left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n\right) = \frac{f^{(n)}(0)}{n!} \times n\cdot(n-1)\cdots 3\cdot 2\cdot 1\cdot x^0 + \text{higher powers of } x$$

which means the n^{th} derivative of Taylor series at $x=a\,(=0)$ is $f^{(n)}(a)$ $(=f^{(n)}(0)$ for Maclaurin series$)$.

This means that if $f(x)$ has a <u>power series representation</u> (at $x=a$), it <u>must</u> be the Taylor series!

<u>E.g.</u> Let's find the Maclaurin series of $f(x) = e^x$.

We know $d/dx(e^x) = e^x$, so in fact $f^{(n)}(x) = e^x$ for all $n \geq 0$, and thus $f^{(n)}(0) = e^0 = 1$ for all $n \geq 0$.

This means the Taylor-Maclaurin series of e^x is

$$\sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

<u>Recall</u>: We saw before that this power series has radius of convergence $R = \infty$.

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<u>WARNING</u>: There is no reason the Taylor series has to converge (i.e., have positive radius of convergence $R>0$), and even if it <u>does</u>, it doesn't necessarily converge to <u>same</u> <u>function</u> as $f(x)$ itself.

<u>E.g.</u> Try $f(x) = \begin{cases} e^{-1/x} & \text{if } x>0 \\ 0 & \text{if } x \le 0 \end{cases}$ as an exercise.

So how to show in practice $f(x)$ equals its Taylor series?

Let us define the degree n <u>Taylor polynomial</u> $T_n(x)$ (centered at $x=a$) of $f(x)$ to be n^{th} partial sum of Taylor series:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

<u>E.g.</u> For $f(x)=e^x$ (and $a=0$), $T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$.

By definition, the Taylor series is $T(x) = \lim_{n\to\infty} T_n(x)$.

So in order to show that $T(x) = f(x)$, in some open interval $|x-a|<d$, we need to look at the <u>remainder</u>

$$R_n(x) = f(x) - T_n(x)$$

and show that $\lim_{n\to\infty} R_n(x) = 0$. To do that...

<u>Theorem</u> (<u>Taylor's Inequality</u>)

Suppose that $|f^{(n+1)}(x)| \le M$ for all $|x-a| \le d$.

Then the remainder for n^{th} Taylor polynomial satisfies

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for all } |x-a| \le d.$$

<u>Note</u>: Notice how we bound the error for $T_n(x)$ in terms of $f^{(n+1)}(x)$, i.e., the <u>next</u> <u>derivative</u> after those appearing in $T_n(x)$.

$\boxed{\text{Skipped!}}$

Let's use Taylor's inequality to show $f(x) = e^x$ is equal to its Taylor-Maclaurin series for all x.
We need to show that for $T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$ and remainder $R_n(x) = f(x) - T_n(x)$, have $\lim_{n\to\infty} R_n(x) = 0$.
Fix an arbitrary d and focus on x where $|x| \le d$.
By Taylor's inequality, have

$$|R_n(x)| \le \frac{M}{(n+1)!} |x|^{n+1}, \text{ where}$$

$|f^{(n+1)}(x)| \le M$ is a bound on the $(n+1)^{st}$ derivative.
But note that for any n, $f^{(n+1)}(x) = e^x$, so a bound on $|f^{(n+1)}(x)|$ is just e^d if $|x| \le d$. Thus,

$$|R_n(x)| \le \frac{e^d}{(n+1)!} |x|^{n+1} \text{ for all } |x| \le d.$$

Hence, $\lim_{n\to\infty} |R_n(x)| \le \lim_{n\to\infty} \frac{e^d}{(n+1)!} |x|^{n+1}$

$$= e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

Where we used the important fact $\boxed{\lim_{n\to\infty} \frac{r^n}{n!} = 0}$
for any fixed r (factorial is "super exponential").
Since the d we fixed was arbitrary, we get

$$\lim_{n\to\infty} R_n(x) = 0 \text{ for all } x,$$

and thus $\boxed{e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots$
for all x.
Key point: This worked because derivative $f^{(n)}(x)$ of $f(x)$ did not increase with n.

More important Taylor series §11.10

Let's find the Taylor-Maclaurin series for $f(x) = \sin(x)$.

To do this, we need to take derivatives of $\sin(x)$:

$$f^{(0)}(x) = \sin(x) \Rightarrow f^{(0)}(0) = 0$$
$$f^{(1)}(x) = \cos(x) \Rightarrow f^{(1)}(0) = 1$$
$$f^{(2)}(x) = -\sin(x) \Rightarrow f^{(2)}(0) = 0$$
$$f^{(3)}(x) = -\cos(x) \Rightarrow f^{(3)}(0) = -1$$

and then $f^{(4)}(x) = \sin(x)$ so this pattern $0, 1, 0, -1$ repeats.

This means $f^{(n)}(0) = \begin{cases} 0 & n \text{ even} \\ (-1)^m & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$

So the Taylor-Maclaurin series of $\sin(x)$ is

$$\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Moreover, because $|f^{(n)}(x)|$ is bounded for all x, the same technique using Taylor's inequality we we used to show that e^x equals its Taylor series for all x works also for $\sin(x)$:

$$\sin(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \text{ for all } x.$$

Something very similar happens for $f(x) = \cos(x)$.

This time the pattern of $f^{(n)}(0)$ is $1, 0, -1, 0, \ldots$

and again $\cos(x)$ equals its Taylor series for all x:

$$\cos(x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Note: Can also find this formula by taking derivative of Taylor series for $\sin(x)$.

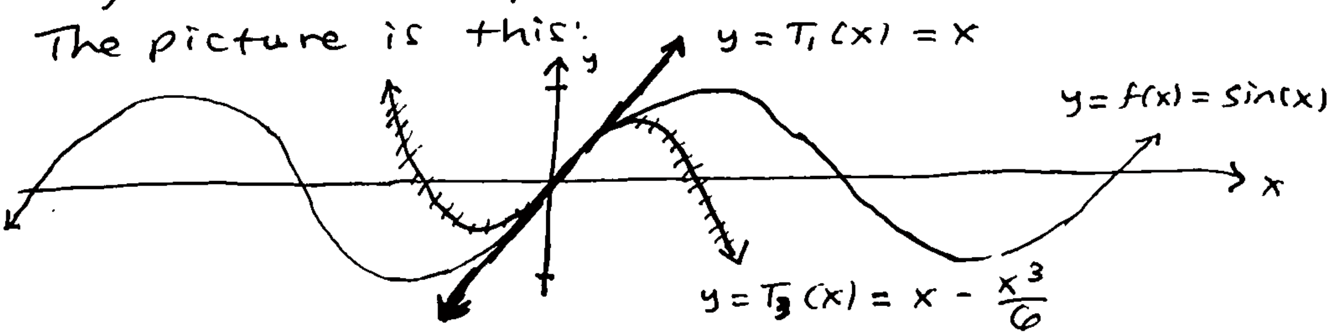
§11.11

Application of Taylor series: approximation

The main application of Taylor series/polynomials is approximation.
If $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is the Taylor series of $f(x)$, and $T(x) = f(x)$ for all $|x-a| < R$ (radius of convergence), then we can expect $f(x) \approx T_n(x)$ for $x \approx a$, where $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the Taylor polynomial

E.g.: To evaluate $\sin(\frac{1}{2})$ we can use the degree 3 Taylor polynomial approximation: $\sin(x) \approx x - \frac{x^3}{6}$, so $\sin(\frac{1}{2}) \approx \frac{1}{2} - \frac{1}{6}(\frac{1}{3})^3 = \frac{1}{2} - \frac{1}{48} \approx 0{,}48\ldots$

To get a better approximation, use a higher value of n.
The picture is this:



Each $T_n(x)$ does a better and better job of approximating $f(x)$ for x near the center a of the Taylor series.
Notice that $y = T_1(x) = f(a) + f'(a)(x-a)$ is the tangent line to curve $y = f(x)$ at $x = a$, best linear approx.
So Taylor polynomials are nonlinear generalizations of the tangent line approximation.
To bound error of our approximations, we use Taylor's inequality or other inequalities (integral test, alternating series test, etc.) //