COMMON CALCULUS 1 INTEGRALS

$$\int e^x dx = e^x + C$$

$$\int b^x dx = \frac{b^x}{\ln b} + C$$

$$\int k dx = kx + C$$

$$\int \cos x dx = \sin x + C$$

$$\int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C$$

$$\int \frac{dx}{|x|\sqrt{x^2-1}} = \sec^{-1} x + C$$

DEFINITE INTEGRAL DEFINITION

$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^n f(x_k)\Delta x$ where $\Delta x = \frac{b-a}{n}$ a $x_k = a + k\Delta x$

FUNDAMENTAL THEOREM OF CALCULUS, PART I

Assume $f(x)$ is continuous on $[a,b]$. If $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$, then $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$

FUNDAMENTAL THEOREM OF CALCULUS, PART II

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x) \qquad \text{(chain rule version)}$$

BASIC INTEGRATION PROPERTIES

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^a f(x)dx = 0$$

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx (a \le b \le c)$$

$$\int_a^b k dx = k(b-a)$$

INTEGRATION BY PARTS

$\int u dv = uv - \int v du$ or

$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$ Remember the acronym **ILATE** when choosing u.

Inverse Trig, **L**ogarithmic, **A**lgebraic, **T**rigonometric, **E**xponential

MORE INTEGRATION PROPERTIES

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$$

If $f(x) \ge 0$ for $a \le x \le b$, then

$$\int_a^b f(x)dx \ge 0$$

If $f(x) \ge g(x)$ for $a \le x \le b$, then

$$\int_a^b f(x)dx \ge \int_a^b g(x)dx$$

If $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

INTEGRATION BY SUBSTITUTION

$\int f(g(x))g'(x)dx = \int f(u)du$ or

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

where $u = g(x)$ and $du = g'(x)dx$

COMMON CALCULUS 2 INTEGRALS

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$$

$$\int \tan x dx = \ln|\sec x| + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x dx = \ln|\sin x| + C$$

$$\int \csc x dx = -\ln|\csc x + \cot x| + C$$

$$\int \ln x dx = x\ln x - x + C$$

$$\int \tan^{-1} x dx = x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + C$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

where $F(x)$ is any antiderivative of $f(x)$ and k is any nonzero constant. For example, $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$ and $\int \sin(kx)dx = -\frac{1}{k}\cos(kx) + C$

ARC LENGTH FORMULA

Arc length: $\int_a^b \sqrt{1+(f'(x))^2}\,dx$ or $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

VOLUMES OF SOLIDS OF REVOLUTION

DISK METHOD: $\int_a^b \pi(\text{ Radius })^2 dx = \int_a^b \pi(R(x))^2 dx$

WASHER METHOD : $\int_a^b \pi\left[\left(\begin{matrix}\text{Outer}\\ \text{Radius}\end{matrix}\right)^2 - \left(\begin{matrix}\text{Inner}\\ \text{Radius}\end{matrix}\right)^2\right]dx = \int_a^b \pi\left((R(x))^2 - (r(x))^2\right)dx$

SHELL METHOD: $\int_a^b 2\pi\left(\begin{smallmatrix}\text{Shell}\\ \text{Radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{Shell}\\ \text{Height}\end{smallmatrix}\right)dx,$ $\quad \int_a^b 2\pi\left(\begin{smallmatrix}\text{Shell}\\ \text{Radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{Shell}\\ \text{Height}\end{smallmatrix}\right)dy$

TRIGONOMETRIC SUBSTITUTION

EXPRESSION	SUBSTITUTION	EVALUATION
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ $dx = a\cos\theta d\theta$	$\sqrt{a^2 - a^2\sin^2\theta}$ $= a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$ $dx = a\sec^2\theta d\theta$	$\sqrt{a^2 + a^2\tan^2\theta}$ $= a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta d\theta$	$\sqrt{a^2\sec^2\theta - a^2}$ $= a\tan\theta$

AREA BETWEEN TWO CURVES

$A = \int_a^b |f(x) - g(x)|\,dx$

VOLUME BY CROSS-SECTIONS (GENERAL SOLIDS)

$V = \int_a^b A(x)\,dx$ or $V = \int_c^d A(y)\,dy$ where $A(x)$ is the area of the cross-section perpendicular to the axis.

AREA OF SURFACE OF REVOLUTION

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

arc length $= \int ds$

surface area $= \begin{cases} \int 2\pi y ds & \text{rotate around x-axis} \\ \int 2\pi x ds & \text{rotate around y-axis} \end{cases}$

ARC LENGTH AND SURFACE AREA (PARAMETRIC)

arc length $= \displaystyle\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

surface area $= \begin{cases} \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt & \text{rotate around } x\text{-axis} \\ \int_\alpha^\beta 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt & \text{rotate around } y\text{-axis} \end{cases}$

AREA, ARC LENGTH, IN POLAR COORDINATES

area $= \displaystyle\int_\alpha^\beta \frac{1}{2} r^2 d\theta$ arc length $= \displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

AVERAGE VALUE OF A FUNCTION

$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$

APPROXIMATING INTEGRALS

$$\text{Left Sum: } L_n = \Delta x \sum_{i=0}^{n-1} f(x_i) \qquad \Delta x = \frac{b-a}{n}$$

$$\text{Right Sum: } R_n = \Delta x \sum_{i=1}^{n} f(x_i)$$

$$\text{Midpoint Rule: } M_n = \Delta x \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)$$

$$\text{TRAPEZOIDAL RULE: } T_n = \frac{\Delta x}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)$$

$$\text{Simpson's Rule: } S_n = \frac{\Delta x}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 4f_{n-1} + f_n\right)$$

(n even)

WORK (FORCE × DISTANCE PROBLEMS)

$W = \int_a^b F(x)\,dx$ $F(x)$: variable force applied over [a,b]

MOMENTS AND CENTERS OF MASS

$M_y = \rho \int_a^b x[f(x) - g(x)]dx,$ $M_x = \rho \int_a^b \frac{1}{2}\left[f^2(x) - g^2(x)\right]dx$

$\bar{x} = \frac{M_y}{\rho A},$ $\bar{y} = \frac{M_x}{\rho A}$

REMAINDER ESTIMATE FOR THE INTEGRAL TEST

$\int_{n+1}^\infty f(x)dx \le R_n \le \int_n^\infty f(x)dx$

$S_n + \int_{n+1}^\infty f(x)dx \le S \le S_n + \int_n^\infty f(x)dx$

SEQUENCES

$\lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$

Monotone $+$ Bounded $\Rightarrow$ convergent

CONVERGENCE TESTS SUMMARY

Geometric: $\sum ar^n$ conv. if $|r| < 1$ $\left(= \frac{a}{1-r}\right)$

p-series: $\sum \frac{1}{n^p}$ conv. $\iff p > 1$

Integral: $\int_a^\infty f(x)\,dx$ conv. $\Longrightarrow \sum f(n)$ conv.

Comparison: $0 \le a_n \le b_n$, $\sum b_n$ conv. $\Rightarrow \sum a_n$ conv.

Limit comp.: $\lim_{n\to\infty} \frac{a_n}{b_n} = L \; (0 < L < \infty) \Rightarrow$ same behaviour

Ratio: $\rho = \lim \left|\frac{a_{n+1}}{a_n}\right|$, $\rho < 1 \Rightarrow$ conv.

Root: $L = \lim \sqrt[n]{|a_n|}$, $L < 1 \Rightarrow$ conv.

Alt. series: $b_n \downarrow\to 0 \Rightarrow \sum(-1)^n b_n$ conv.

Absolute: $\sum |a_n|$ conv. $\Rightarrow \sum a_n$ conv.

TAYLOR SERIES

$$f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

SOME MACLAURIN SERIES AND INTERVAL OF CONVERGENCE

Function	Series	Interval of conv.
$\frac{1}{1-x}$	$= \sum_{n=0}^\infty x^n = 1 + x + x^2 + x^3 + \cdots$	$(-1,1)$
e^x	$= \sum_{n=0}^\infty \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$(-\infty,\infty)$
$\sin x$	$= \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	$(-\infty,\infty)$
$\cos x$	$= \sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	$(-\infty,\infty)$
$\ln(1+x)$	$= \sum_{n=1}^\infty (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1,1]$
$\tan^{-1} x$	$= \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$	$[-1,1]$
$(1+x)^m$	$= \sum_{n=1}^\infty \binom{m}{n} x^k = 1 + mx + \frac{m(m-1)x^2}{2!} + \cdots$	$(-1,1)$