Final Exam Study Guide
Math 157 (Calculus II), Spring 2024

1. Geometric applications of integrals [§6.1, 6.2, 6.3, 8.1, 8.2]
   (a) Area between curves [§6.1]: area between $y = f(x)$ and $y = g(x)$ is $\int_a^b |f(x) - g(x)|\, dx$.
   (b) Volume of general solid [§6.2]: if $A(x) =$ area of cross-section, then volume is $\int_a^b A(x)\, dx$.
   (c) Volume of solid of revolution [§6.2, 6.3]: "disks/washers" & "cylindrical shells" methods. For region below curve $y = f(x)$ from $x = a$ to $x = b$:
      i. rotated around x-axis, "disks method" gives volume $= \int_a^b \pi\, f(x)^2\, dx$;
      ii. rotated around y-axis, "shells method" gives volume $= \int_a^b 2\pi\, f(x)\, x\, dx$.
   (d) Arc lengths of curves [§8.1]: length of $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$.
   (e) Area of surface of revolution [§8.2]:
      i. for $y = f(x)$ from $x = a$ to $x = b$ rotated about x-axis, area is $\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx$;
      ii. for $x = g(y)$ from $y = c$ to $y = d$ rotated about x-axis, area is $\int_c^d 2\pi y \sqrt{1 + (g'(y))^2}\, dy$.
2. Other applications of integrals [§6.4, 6.5]
   (a) Work [§6.4]: if $F(x) =$ force as function of distance, then work done is $W = \int_a^b F(x)\, dx$.
   (b) Average of function [§6.5]: the average of $f(x)$ from $x = a$ to $x = b$ is $\frac{1}{b-a} \int_a^b f(x)\, dx$.
3. Techniques for computing integrals [§7.1, 7.2, 7.3, 7.4, 7.5]
   (a) Integration by parts [§7.1]: $\int u\, dv = uv - \int v\, du$; choose u using "LIATE" rule
   (b) Trigonometric integrals [§7.2]: for $\int \sin^n(x) \cos^m(x)\, dx$, use the Pythagorean identity $\sin^2(x) + \cos^2(x) = 1$ to isolate single factor of $\cos(x)\, dx$ or $\sin(x)\, dx$, then do a u-sub.
   (c) Trigonometric substitution [§7.3]:
      i. for $a^2 - x^2 \Rightarrow$ sub $x = a\sin(\theta)$, $dx = a\cos(\theta)\, d\theta$, and use $1 - \sin^2(\theta) = \cos^2(\theta)$;
      ii. for $a^2 + x^2 \Rightarrow$ sub $x = a\tan(\theta)$, $dx = a\sec^2(\theta)\, d\theta$, and use $1 + \tan^2(\theta) = \sec^2(\theta)$.
   (d) Integrating rational functions by partial fractions [§7.4]: find roots of denominator $Q(x)$ and solve system of equations to write $P(x)/Q(x) = A/(x-a) + B/(x-b) + ... + Z/(x-z)$ and use $\int A/(x-a)\, dx = A\ln(x-a)$; for repeated roots do $A_1/(x-a) + A_2/(x-a)^2 + \cdots$.
4. Other concepts related to integration [§7.7, 7.8]
   (a) Approximating definite integrals [§7.7]: two good approximations of $\int_a^b f(x)\, dx$ are
      i. midpoint approximation $M_n = \sum_{i=1}^n f(\overline{x}_i)\Delta x$ where $\overline{x}_i = \frac{x_{i-1}+x_i}{2}$;
      ii. trapezoid approximation $T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))$.
   (b) Improper integrals [§7.8]: $\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx$, et cetera.

5. Parametrized curves [§10.1, 10.2]
   (a) Curve of form $x = f(t)$ and $y = g(t)$ for some auxiliary variable t ("time") [§10.1]
   (b) Slope of tangent [§10.2] to curve given by chain rule: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$
   (c) Arc length [§10.2] is $\int_a^b \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}\, dt = \int_a^b \sqrt{g'(t)^2 + f'(t)^2}\, dt$
6. Polar coordinates and polar curves [§10.3, 10.4]
   (a) Cartesian vs. polar [§10.3]: $(x, y) = (r\cos\theta, r\sin\theta)$ and $(r, \theta) = (\sqrt{x^2+y^2}, \arctan(\frac{y}{x}))$
   (b) Area inside [§10.4] polar curve $r = f(\theta)$ for $\alpha \le \theta \le \beta$ is $\int_\alpha^\beta \frac{1}{2} r^2\, d\theta = \int_\alpha^\beta \frac{1}{2} f(\theta)^2\, d\theta$
   (c) Slope of tangent [§10.4] to polar curve $r = f(\theta)$ given by chain and product rules:
   $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{f(\theta)\,\cos\theta + f'(\theta)\,\sin\theta}{f'(\theta)\cos\theta - f(\theta)\,\sin\theta}$$
   (d) Arc length [§10.4] of polar curve $r = f(\theta)$ is $\int_\alpha^\beta \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$
7. Sequences and series [§11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7]
   (a) Sequence $\{a_n\}_{n=1}^\infty = a_1, a_2, \ldots$ is list of numbers, $\lim\limits_{n\to\infty} a_n$ defined like $\lim\limits_{x\to\infty} f(x)$ [§11.1]
   (b) Series $\sum_n^\infty a_n$ is "infinite sum" $a_1 + a_2 + \cdots$ of terms a_n; its value is $s = \lim_{n\to\infty} s_n$ where $s_n = a_1 + a_2 + \cdots + a_n$ is the nth partial sum [§11.2]
   (c) Important series: geometric series [§11.2] $\sum_{n=1}^\infty ar^{n-1}$ converges if and only if $|r| < 1$ (and $= \frac{a}{1-r}$ if it converges); p-series [§11.3] $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$
   (d) Many tests for convergence / divergence of series:
      i. (Divergence test [§11.2]) If $\lim_{n\to\infty} a_n \neq 0$, series $\sum_n^\infty a_n$ diverges.
      ii. (Integral test [§11.3]) If $f(x)$ continuous, decreasing, and positive, with $a_n = f(n)$, then $\sum_n^\infty a_n$ converges if and only if $\int_1^\infty f(x)\, dx$ converges. In this case, have error bounds for remainder $R_n = s - s_n$: $\int_{n+1}^\infty f(x)\, dx \le R_n \le \int_n^\infty f(x)\, dx$.
      iii. (Comparison tests [§11.4]) If $\sum_n^\infty b_n$ converges & $a_n \le b_n$, then $\sum_n^\infty a_n$ converges. If $\sum_n^\infty b_n$ diverges & $a_n \ge b_n$, then $\sum_n^\infty a_n$ diverges. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is $\neq 0$, then $\sum_n^\infty a_n$ converges if and only if $\sum_n^\infty b_n$ converges.
      iv. (Alternating series test [§11.5]) Alternating series $\sum_{n=1}^\infty (-1)^{n-1} b_n$ converges as long as $b_{n+1} \le b_n$ and $\lim_{n\to\infty} b_n = 0$. In this case, have error bound: $|R_n| \le b_{n+1}$.
      v. (Ratio test [§11.6]) For series $\sum_{n=1}^\infty a_n$, let $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If $L < 1$, series converges. If $L > 1$ (including ∞), series diverges. If $L = 1$, test is inconclusive.
8. Power series and Taylor series [§11.8, 11.9, 11.10, 11.11]
   (a) The ratio test tells us that any power series $\sum_{n=0}^\infty c_n (x-a)^n$ has a radius of convergence R such that it converges when $|x - a| < R$ and diverges when $|x - a| > R$ [§11.8]
   (b) Power series representations of functions $f(x) = \sum_{n=0}^\infty c_n (x-a)^n$; getting a representation for one function from another via algebraic manipulations (like substitution) [§11.9]
   (c) Differentiate, integrate, and multiply power series like they are polynomials [§11.9, 11.10]
   (d) Taylor series of $f(x)$ at $x = a$ is $\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}$ is nth derivative [§11.10]
   (e) Important Taylor series [§11.10]: $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$ $(R = 1)$; $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}$ $(R = \infty)$; $\sin(x) = \sum_{n=0}^\infty \frac{(-1)^{n-1} x^{2n+1}}{(2n+1)!}$ $(R = \infty)$; $\cos(x) = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}$ $(R = \infty)$
   (f) Taylor polynomial $T_n(x)$: nth partial sum of series; $f(x) \approx T_n(x)$ if $x \approx a$ [§11.10, 11.11]