## Final Exam Study Guide Math 157 (Calculus II), Spring 2025

- 1. Geometric applications of integrals [§6.1, 6.2, 6.3, 8.1, 8.2]
  - (a) Area between curves [§6.1]: area between y = f(x) and y = g(x) is  $\int_a^b |f(x) g(x)| dx$ .
  - (b) Volume of general solid [§6.2]: if A(x) = area of cross-section, then volume is  $\int_a^b A(x) dx$ .
  - (c) Volume of solid of revolution [§6.2, 6.3]: "disks/washers" & "cylindrical shells" methods. For region below curve y = f(x) from x = a to x = b:
    - i. rotated around x-axis, "disks method" gives volume =  $\int_a^b \pi f(x)^2 dx$ ;
    - ii. rotated around *y*-axis, "shells method" gives volume =  $\int_a^b 2\pi f(x) x \, dx$ .
  - (d) Arc lengths of curves [§8.1]: length of y = f(x) from x = a to x = b is  $\int_a^b \sqrt{1 + (f'(x))^2} dx$ .
  - (e) Area of surface of revolution [§8.2]:
    - i. for y = f(x) from x = a to x = b rotated about x-axis, area is  $\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$ ;
    - ii. for x = g(y) from y = c to y = d rotated about x-axis, area is  $\int_{c}^{d} 2\pi y \sqrt{1 + (g'(y))^2} \, dy$ .
- 2. Other applications of integrals [§6.4, 6.5]
  - (a) Work [§6.4]: if F(x) = force as function of distance, then work done is  $W = \int_a^b F(x) dx$ .
  - (b) Average of function [§6.5]: the average of f(x) from x = a to x = b is  $\frac{1}{b-a} \int_a^b f(x) dx$ .
- 3. Techniques for computing integrals [§7.1, 7.2, 7.3, 7.4, 7.5]
  - (a) Integration by parts [§7.1]:  $\int u \, dv = uv \int v \, du$ ; choose u using "LIATE" rule
  - (b) Trigonometric integrals [§7.2]: for  $\int \sin^n(x) \cos^m(x) dx$ , use the Pythagorean identity  $\sin^2(x) + \cos^2(x) = 1$  to isolate single factor of  $\cos(x) dx$  or  $\sin(x) dx$ , then do a *u*-sub.
  - (c) Trigonometric substitution [§7.3]:
    - i. for  $a^2 x^2 \Rightarrow \sup x = a \sin(\theta)$ ,  $dx = a \cos(\theta) d\theta$ , and use  $1 \sin^2(\theta) = \cos^2(\theta)$ ;
    - ii. for  $a^2 + x^2 \Rightarrow \operatorname{sub} x = a \tan(\theta)$ ,  $dx = a \sec^2(\theta) d\theta$ , and use  $1 + \tan^2(\theta) = \sec^2(\theta)$ .
  - (d) Integrating rational functions by partial fractions [§7.4]: find roots of denominator Q(x) and solve system of equations to write P(x)/Q(x) = A/(x-a) + B/(x-b) + ... + Z/(x-z) and use  $\int A/(x-a) dx = A \ln(x-a)$ ; for repeated roots do  $A_1/(x-a) + A_2/(x-a)^2 + \cdots$ .
- 4. Other concepts related to integration [§7.7, 7.8]
  - (a) Approximating definite integrals [§7.7]: two good approximations of  $\int_a^b f(x) dx$  are
    - i. midpoint approximation  $M_n = \sum_{i=1}^n f(\overline{x}_i) \Delta x$  where  $\overline{x}_i = \frac{x_{i-1} + x_i}{2}$ ;
    - ii. trapezoid approximation  $T_n = \frac{\Delta x}{2} (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)).$
  - (b) Improper integrals [§7.8]:  $\int_a^{\infty} f(x) dx = \lim_{t \to \infty} \int_a^t f(x) dx$ , et cetera.

- 5. Parametrized curves [§10.1, 10.2]
  - (a) Curve of form x = f(t) and y = g(t) for some auxiliary variable t ("time") [§10.1]
  - (b) Slope of tangent [§10.2] to curve given by chain rule:  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$
  - (c) Arc length [§10.2] is  $\int_a^b \sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2} dt = \int_a^b \sqrt{g'(t)^2 + f'(t)^2} dt$

6. Polar coordinates and polar curves [§10.3, 10.4]

- (a) Cartesian vs. polar [§10.3]:  $(x, y) = (r \cos \theta, r \sin \theta)$  and  $(r, \theta) = (\sqrt{x^2 + y^2}, \arctan(\frac{y}{r}))$
- (b) Area inside [§10.4] polar curve  $r = f(\theta)$  for  $\alpha \le \theta \le \beta$  is  $\int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta$
- (c) Slope of tangent [§10.4] to polar curve  $r = f(\theta)$  given by chain and product rules:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

(d) Arc length [§10.4] of polar curve  $r = f(\theta)$  is  $\int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta$ 

- 7. Sequences and series [§11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7]
  - (a) Sequence  $\{a_n\}_{n=1}^{\infty} = a_1, a_2, \dots$  is list of numbers,  $\lim_{n \to \infty} a_n$  defined like  $\lim_{x \to \infty} f(x)$  [§11.1]
  - (b) Series  $\sum_{n=1}^{\infty} a_n$  is "infinite sum"  $a_1 + a_2 + \cdots$  of terms  $a_n$ ; its value is  $s = \lim_{n \to \infty} s_n$ where  $s_n = a_1 + a_2 + \dots + a_n$  is the *n*th partial sum [§11.2]
  - (c) Important series: geometric series [§11.2]  $\sum_{n=1}^{\infty} ar^{n-1}$  converges if and only if |r| < 1(and  $= \frac{a}{1-r}$  if it converges); *p*-series [§11.3]  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if and only if p > 1
  - (d) Many tests for convergence / divergence of series:
    - i. (Divergence test [§11.2]) If  $\lim_{n\to\infty} a_n \neq 0$ , series  $\sum_n^{\infty} a_n$  diverges.
    - ii. (Integral test [§11.3]) If f(x) continuous, decreasing, and positive, with  $a_n = f(n)$ , then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\int_1^{\infty} f(x) dx$  converges. In this case, have error bounds for remainder  $R_n = s - s_n$ :  $\int_{n+1}^{\infty} f(x) dx \le R_n \le \int_n^{\infty} f(x) dx$ .
    - iii. (Comparison tests [§11.4] for series w/ positive terms) If  $\sum_{n=0}^{\infty} b_n$  converges &  $a_n \leq b_n$ , then  $\sum_{n=1}^{\infty} a_n$  converges. If  $\sum_{n=1}^{\infty} b_n$  diverges &  $a_n \ge b_n$ , then  $\sum_{n=1}^{\infty} a_n$  diverges. If  $\lim_{n\to\infty} a_n/b_n \ne 0, \pm\infty$ , then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} b_n$  converges.
    - iv. (Alternating series test [§11.5]) Alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$  converges as long as  $b_{n+1} \leq b_n$  and  $\lim_{n \to \infty} b_n = 0$ . In this case, have error bound:  $|R_n| \leq b_{n+1}$ .
    - v. (Ratio test [§11.6]) For series  $\sum_{n=1}^{\infty} a_n$ , let  $L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$ . If L < 1, series converges. If L > 1 (including  $\infty$ ), series diverges. If L = 1, test is inconclusive.
- 8. Power series and Taylor series [§11.8, 11.9, 11.10, 11.11]
  - (a) The ratio test tells us that any power series  $\sum_{n=0}^{\infty} c_n (x-a)^n$  has a radius of convergence R such that it converges when |x - a| < R and diverges when |x - a| > R [§11.8]
  - (b) Power series representations of functions  $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ ; getting a representation for one function from another via algebraic manipulations (like substitution) [§11.9]
  - (c) Differentiate, integrate, and multiply power series like they are polynomials [§11.9, 11.10]

  - (d) Taylor series of f(x) at x = a is  $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ , where  $f^{(n)}$  is nth derivative [§11.10] (e) Important Taylor series [§11.10]:  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \ (R=1); \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \ (R=\infty);$   $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{(2n+1)!} \ (R=\infty); \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \ (R=\infty)$
  - (f) Taylor polynomial  $T_n(x)$ : nth partial sum of series;  $f(x) \approx T_n(x)$  if  $x \approx a$  [§11.10, 11.11]