Midterm #3 Study Guide
Math 157 (Calculus II), Spring 2024

1. Sequences and series [§11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7]
   (a) Sequence $\{a_n\}_{n=1}^{\infty} = a_1, a_2, \ldots$ is list of numbers, $\lim\limits_{n\to\infty} a_n$ defined like $\lim\limits_{x\to\infty} f(x)$ [§11.1]
   (b) Series $\sum_n^{\infty} a_n$ is "infinite sum" $a_1 + a_2 + \cdots$ of terms a_n; its value is $s = \lim_{n\to\infty} s_n$ where $s_n = a_1 + a_2 + \cdots + a_n$ is the nth partial sum [§11.2]
   (c) Important series: geometric series [§11.2] $\sum_{n=1}^{\infty} ar^{n-1}$ converges if and only if $|r| < 1$ (and $= \frac{a}{1-r}$ if it converges); p-series [§11.3] $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$
   (d) Many tests for convergence / divergence of series:
      i. (Divergence test [§11.2]) If $\lim_{n\to\infty} a_n \neq 0$, series $\sum_n^{\infty} a_n$ diverges.
      ii. (Integral test [§11.3]) If $f(x)$ continuous, decreasing, and positive, with $a_n = f(n)$, then $\sum_n^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\,dx$ converges. In this case, have error bounds for remainder $R_n = s - s_n$: $\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$.
      iii. (Comparison tests [§11.4]) If $\sum_n^{\infty} b_n$ converges & $a_n \leq b_n$, then $\sum_n^{\infty} a_n$ converges. If $\sum_n^{\infty} b_n$ diverges & $a_n \geq b_n$, then $\sum_n^{\infty} a_n$ diverges. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is $\neq 0$, then $\sum_n^{\infty} a_n$ converges if and only if $\sum_n^{\infty} b_n$ converges.
      iv. (Alternating series test [§11.5]) Alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges as long as $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$. In this case, have error bound: $|R_n| \leq b_{n+1}$.
      v. (Ratio test [§11.6]) For series $\sum_{n=1}^{\infty} a_n$, let $L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If $L < 1$, series converges. If $L > 1$ (including ∞), series diverges. If $L = 1$, test is inconclusive.

2. Power series and Taylor series [§11.8, 11.9, 11.10, 11.11]
   (a) The ratio test tells us that any power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius of convergence R such that it converges when $|x-a| < R$ and diverges when $|x-a| > R$ [§11.8]
   (b) Power series representations of functions $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$; getting a representation for one function from another via algebraic manipulations (like substitution) [§11.9]
   (c) Differentiate, integrate, and multiply power series like they are polynomials [§11.9, 11.10]
   (d) Taylor series of $f(x)$ at $x = a$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, where $f^{(n)}$ is nth derivative [§11.10]
   (e) Important Taylor series [§11.10]: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $(R = 1)$; $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $(R = \infty)$; $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}x^{2n+1}}{(2n+1)!}$ $(R = \infty)$; $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $(R = \infty)$
   (f) Taylor polynomial $T_n(x)$: nth partial sum of series; $f(x) \approx T_n(x)$ if $x \approx a$ [§11.10, 11.11]