Howard Math 181: Discrete Structures Spring 2024

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(call me "Sam")

1/9 Logistics:

Classes: MWF 3:10-4pm, Douglass Hall #113/114

Office hrs: Wed 2-3pm, Annex III - #220
or by appointment - email me!

Website: samuelfhopkins.com/classes/181.html

Text: Discrete Mathematics by Johnsonbaugh, 8e

Grading: 40% homeworks
40% two (in-person) midterms
20% final exam

There will be 12 homeworks, assigned on Wednesdays, and due the following Wednesday in-class! Your lowest 2 scores will be dropped (so 10/12 will count)

The 2 midterms will happen in-class on Wednesdays

The final will take place during finals week

Beyond that, I expect you to
SHOW UP TO CLASS
+ PARTICIPATE! :)

That means... interrupt me by
ASKING QUESTIONS!

(and please say your name when you ask a question, so I learn to put names to faces...)

So... what is "discrete math"?

Discrete	Continuous
• • • • • •	
finite	infinite
integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	real numbers $\mathbb{R} = \{\ldots, 0, \frac{1}{3}, \pi, -e, \ldots\}$
$\cdots -2\ -1\ \ 0\ \ 1\ \ 2\ \ 3 \cdots$	$-e\ \ 0\ \ \frac{1}{3}\ \pi$
algebra (ish...)	calculus
Computer science	(classical) physics

The main topics we will cover are:

- Basic mathematical structures: sets, functions, sequences, relations — Ch's 1 + 3
- Logic and proofs — Ch's 1 + 2
- Basic combinatorics (a.k.a. counting!) — Ch. 5

A kind of problem you should be able to solve by the end is...

["If N people are in a room, and each person shakes everyone else's hand once, how many handshakes happen?"]

But... another major goal of the course is for you to learn how to write mathematical proofs which means convincing, logical arguments.

So the point will be not just to get the right answer/formula, but to be able to explain why your answer is correct!

Sets (§ 1.1 of textbook):

We will start by reviewing sets, the most basic kind of mathematical object. You probably have already seen sets in calculus...

A **set** is just any collection of objects. For example, the collection of all the planets in the solar system forms a set.

We use brackets to denote sets, so that set is...!

No pluto :(→ {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}

The objects that belong to a set are called its elements.

So Mercury is an element of the set of planets

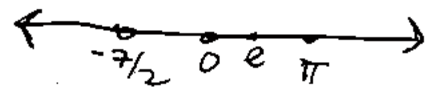
Often we work with sets of numbers. For example, $A = \{1, 2, 3\}$ is a set of three numbers. $B = \{2, 5, 9\}$ is another set of three numbers. We have $2 \in A$ and $2 \in B$ where $\in$ = "is an element of"

Some sets of numbers you know about are

the integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
("Zahlen" = "number" in German)

the rationals $\mathbb{Q} = \{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \}$
("quotients")

the real numbers $\mathbb{R}$ =



Note that these are all infinite sets...

To define $\mathbb{Q}$ above we used set-builder notation:
Notation $\{x : \text{condition on } x\}$ means the set of all x's satisfying this condition
(Note the book writes $\{x \mid \text{condition on } x\}$)

E.g. $\{x : x > 0, x \in \mathbb{Z}\} = \{1, 2, 3, \ldots\}$

1/11 Q: What is $\{x : x^2 = 1, x \in \mathbb{R}\}$?
A: $\{-1, 1\}$ since $(-1)^2 = 1$ and $1^2 = 1$
(and these are the only #'s squaring to one...)

There is a special set, called the empty set (or null set) and denoted $\emptyset$ (or $\{\}$) that has no elements.

Q: What is $\{x : x^2 = -1, x \in \mathbb{R}\}$?
A: The empty set $\emptyset$, since no real numbers square to negative one ($x^2 \geq 0$ for all $x \in \mathbb{R}$).

We say that set A is a subset of a set B if every element of A is also an element of B.
E.g. $\{2, 5\}$ is a subset of $\{2, 3, 5, 10\}$
E.g. $\mathbb{Z}$ is a subset of $\mathbb{Q}$, which is a subset of $\mathbb{R}$
We use $\subseteq$ to denote "is a subset of"
so $\{1, 2\} \subseteq \{1, 2, 3, 4\}$ and $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

The set of all subsets of a set A is called the <u>power set</u> of A, and is denoted $P(A)$.

<u>E.g.</u> If $A = \{a, b, c\}$ its power set is

$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

<u>Note</u>: A has 3 elements and its power set has $2^3 = 8$ elements.

We use $|A|$ (or $\#A$) to denote the number of elements of a finite set A. In the example above, we have $|A| = 3$ and $|P(A)| = 2^3 = 8$.

Later we'll show that $|P(A)| = 2^{|A|}$ for all finite sets A.

Notice that the empty set $\emptyset$ is a subset of <u>every</u> set A. Also, A is always a subset of itself. In symbols:

$\emptyset \subseteq A$ and $A \subseteq A$ for all A.

These two subsets are called <u>trivial</u> <u>subsets</u> of A, and the other (nontrivial) subsets are called the <u>proper</u> <u>subsets</u>.

<u>E.g.</u> The proper subsets of $A = \{a, b, c\}$ are $\{a\}$, $\{b\}$, $\{c\}$, $\{a,b\}$, $\{a,c\}$ and $\{b,c\}$

(There are $2^3 - 2 = 8 - 2 = 6$ proper subsets of this A).

Operations on sets

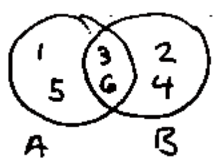
There are various ways to make new sets from old sets.
Given two sets A and B, their union $A \cup B$ is
$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ (or both!)}\}$$
and their intersection $A \cap B$ is
$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$
E.g. If $A = \{1, 3, 5, 6\}$ and $B = \{2, 3, 4, 6\}$ then
$A \cup B = \{1, 2, 3, 4, 5, 6\}$ and $A \cap B = \{3, 6\}$.

The (set) difference of B from A (or "A minus B") is

book uses $A - B$ for this $\rightarrow$ $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$ ← not in

E.g. w/ A and B as above, $A \setminus B = \{1, 5\}$
and $B \setminus A = \{2, 4\}$.

It is convenient to use Venn diagrams to represent the relations between sets, unions, intersections, etc.:

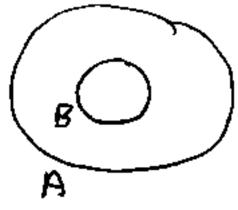


← Venn diagram has the elements of a set inside circle labeled by that set

Then: we can represent

$A \cap B$ intersection — $A \cup B$ union — $A \setminus B$ difference

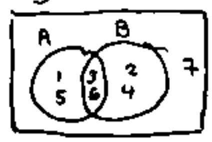
and can also represent subset relation: using Venn diagrams



← means $B \subseteq A$
B is a subset of A

Sometimes there is a universal set $\mathcal{U}$ around, with all sets being a subset of this $\mathcal{U}$.

We draw that using Venn diagrams: like this



$\mathcal{U} = \{1,2,3,4,5,6,7\}$

Book writes $\bar{A}$ for complement → The complement of A is $A^c = \mathcal{U} \setminus A$, ← things not in A. where the universe $\mathcal{U}$ is understood from context.

E.g. In example above, $A^c = \{2,4,7\}$ and $(A \cup B)^c = \{7\}$.

There are many rules that $\cup$, $\cap$, ^{c}, etc. satisfy. Some of the most important of these are:

Theorem (0) Symmetry of $\cup$ and $\cap$: $A \cup B = B \cup A$, $A \cap B = B \cap A$

(0') Involutive behavior of ^{c}: $(A^c)^c = A$

(1) Associativity of $\cup$ and $\cap$:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(2) Distributivity of $\cup$ over $\cap$ and $\cap$ over $\cup$:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

[Think of how $a \times (b+c) = (a \times b) + (a \times c)$]

(3) DeMorgan's Laws:

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

Exercise: Think about the meaning of these rules using Venn diagrams.

We will discuss proofs of these rules at a later point in the course...

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<u>More discrete structures related to sets</u>

A <u>partition</u> of a set A is a collection $\{S_1, S_2, \ldots, S_k\}$ of nonempty <u>subsets</u> $\emptyset \subsetneq S_1, S_2, \ldots, S_k \subseteq A$ such that

- they are pairwise <u>disjoint</u>, meaning $S_i \cap S_j = \emptyset$ for all $i \neq j$ ↪ Venn diagram of disjoint sets:  A B
- their union $S_1 \cup S_2 \cup \cdots \cup S_k = A$ is <u>all</u> of A.

Less formally, a partition is a way of <u>breaking up</u> a set A into (nonempty) subsets $S_1, \ldots, S_k$ so that every element $x \in A$ belongs to a <u>unique</u> one of the subsets $S_1, \ldots, S_k$.

<u>E.g.</u> If $A = \{1,2,3,4,5\}$ then one partition of A is $\{\{1,2,4\},\{3,5\}\}$.
Another one is $\{\{1,5\},\{2,4\},\{3\}\}$.

Because writing so many brackets can be cumbersome, we sometimes use a shorthand where the parts of a partition are <u>divided</u> by <u>vertical lines</u>, like:

$1,2,4 \mid 3,5$ or $1,5 \mid 2,4 \mid 3$

Another way to think of a partition is as a way of <u>grouping together</u> elements of a set into different parts.

<u>E.g:</u> A partition of {People who live in USA} is: People in Alabama | People in Alaska | … | Ppl in Wyoming | ppl in DC, P.R., & other territories

Later when we learn about <u>relations</u> we will see that partitions are closely connected to <u>equivalence</u> <u>relations</u> …

A set is by definition an unordered collection, so that $\{1,2,3\} = \{2,1,3\} = \{3,2,1\} = \cdots$ etc.

(and also we don't care about "multiplicities" So... $\{1,1,2,2,2,3\} \text{ "="} \{1,2,3\}$)

But sometimes we do want to keep track of order.

An ordered pair is an object of the form (a,b), which is considered distinct from (b,a) (if $a \neq b$).

For two sets X and Y, the set of all ordered pairs of the form (x,y) with $x \in X$ and $y \in Y$ is called their (Cartesian) product, denoted $X \times Y$.

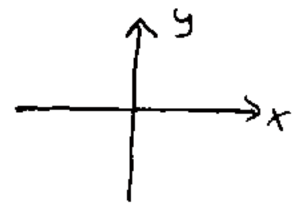
E.g. If $X = \{1,2,3\}$ and $Y = \{a,b\}$ then

$$X \times Y = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}$$
$$Y \times X = \{(a,1),(b,1),(a,2),(b,2),(a,3),(b,3)\}$$
$$Y \times Y = \{(a,a),(a,b),(b,a),(b,b)\}, \text{ etc...}$$

E.g. If $X = \mathbb{R}$ real numbers, then

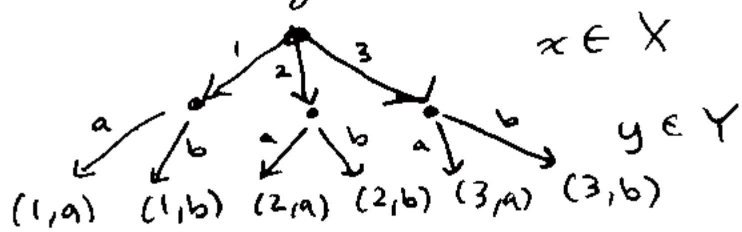
$X \times X = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\} \rightarrow$

"Cartesian plane" / "Cartesian coordinates"



Thm If X and Y are finite, then $|X \times Y| = |X| \cdot |Y|$.

Proof: Imagine constructing an ordered pair (x,y) by first choosing $x \in X$ and then choosing $y \in Y$:



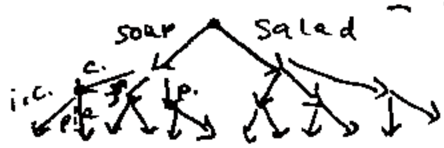
This <u>decision tree</u> will have $|X|$ branches at 1^{st} level, and each of those branches will break into $|Y|$ branches at 2^{nd} level, giving $|X|\cdot|Y|$ total endpoints ("<u>leaves</u>"), which correspond to all the elements of $X \times Y$. ∎

We don't have to stop at two elements. An <u>ordered n-tuple</u> is something of the form $(x_1, x_2, \dots, x_n)$ (considered distinct from all permutations) and for sets $X_1, \dots, X_n$ we let $X_1 \times X_2 \times \dots \times X_n = \{(x_1, \dots, x_n) : \underset{\text{for all } i}{x_i \in X_i}\}$.

<u>E.g.</u> If $X = \{\text{soup, salad}\}$, $Y = \{\text{chicken, fish, pasta}\}$ and $Z = \{\text{ice cream, pie}\}$ then $X \times Y \times Z = \{\text{3-course meals}\}$ with one element being $(\text{salad, fish, pie}) \in X \times Y \times Z$.

<u>Thm</u> $|X_1 \times X_2 \times \dots \times X_n| = |X_1| \cdot |X_2| \cdot \dots \cdot |X_n|$.

<u>Pf</u>: Imagine making a decision tree with n <u>layers</u>: In each layer, branches break into $|X_i|$ new branches, so that



in the end there will be $|X_1| \cdot |X_2| \cdot \dots \cdot |X_n|$ total leaves. ∎

<u>Exercise</u>: Use a decision tree to show why $|P(A)| = 2^{|A|}$ for any finite set A.

<u>Hint</u>: Think of building a subset of A by <u>including</u> or <u>excluding</u> each element $x \in A$ one-by-one ...

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§1.2 Propositions: We've discussed sets for a while. Now we will start a new topic: logic.
The basic things we analyze in logic are propositions.
A proposition is a statement that can be either true or false but not both.

E.g.: (a) The boiling point of water at sea level is 100° C.
(b) August has only 30 days in it.
(c) There is life on Mars.
(d) Take Calculus III next semester!
(e) $4 + x = 6$
(f) The positive integers dividing 7 are 1 and 7.

(a), (b), (c), and (f) are propositions. (a) + (f) are true.
(b) is false (August has 31 days). (c) is either
but not both → true or false, even though we don't know which.
(d) is not a proposition because it's not a statement ("it's a command!")
(e) is not a proposition because it is sometimes true (for $x = 2$) and sometimes false (for other x). [It is a formula... we will discuss these later.]

We use lowercase letters like p and q to denote propositions. We also use the notation:

$$p: \quad 1 + 1 = 3$$

to mean that p is the proposition that $1+1=3$ (which is false!)

Just like with sets and the operations of $\cup, \cap$, etc., there are various logical operations we can use to make new propositions from old ones...

<u>Def'n</u> If p, q are two propositions then we write

$p \wedge q$: p <u>and</u> q ("<u>conjunction</u>")

$p \vee q$: p <u>or</u> q ("<u>disjunction</u>")
(or both!) —— "inclusive or"

<u>E.g.</u> p: It is raining, and q: I have an umbrella
then $p \wedge q$: It is raining and I have an umbrella.

p: It's raining, q: I have an umbrella, r: I have a jacket:
$p \wedge (q \vee r)$: It's raining and I have an umbrella or a jacket
(or both...)

We can represent compound propositions via <u>truth tables</u>:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

← truth tables show for all possible truth values of p and q what the truth value of compound prop. is

<u>Def'n</u> If p is a proposition, then
$\neg p$: <u>not</u> p ("<u>negation</u>")
(also sometimes $!p$)

p	$\neg p$
T	F
F	T

By combining $\wedge$, $\vee$ and $\neg$ can make many more propositions.

<u>Q</u>: How to write the <u>exclusive or</u> (XOR) of p and q?

XOR(p, q): either p or q but not both
: $(p \vee q) \wedge (\neg(p \wedge q))$

p	q	$(p \vee q) \wedge (\neg(p \wedge q))$
T	T	F
T	F	T
F	T	T
F	F	F

↙ can check this is right definition by writing the truth table ...

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§1.3 Conditionals

Consider the statement

"If I'm teaching class today, then I'll go to campus."

This is what we call in logic a conditional.

Def'n Given prop's p and q, we define the conditional prop

$p \to q$: if p then q ("p implies q")

In $p \to q$, p is called the hypothesis (or "antecedant") and q is called the conclusion (or "consequent").

When is $p \to q$ true? Let's analyze

p = "I'm teaching class today," q = "I'll go to campus"

- If I'm teaching class and I go to campus, then $p \to q$ is true.
- If I'm teaching and I don't go to campus, then $p \to q$ is false.

But what about if I'm not teaching class?

- If I'm not teaching and I don't go to campus, $p \to q$ is true.
- If I'm not teaching and I do go to campus, $p \to q$ is still true.

This is because the conditional $p \to q$ makes no claim about what happens if p is false.

Thus, the truth table of $p \to q$ is:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$p \to q$ is true if whenever p is true, then q is true (but if p is false, who knows about q?)

Notice that $p \to q$ is <u>not</u> the same as $q \to p$:
"If I'm teaching, then I go to campus" is <u>true</u>
But "If I go to campus, then I'm teaching" is <u>false</u>
(maybe I went to my office to print something, etc...)

The proposition $q \to p$ is called the <u>converse</u> of $p \to q$.
Don't mix up a statement and its converse!

Another way to think about conditionals is in terms of <u>necessary</u> and <u>sufficient</u> <u>conditions</u>.

If q is a <u>necessary condition</u> for p to be true, then $p \to q$.
<u>E.g.</u> Since it is <u>necessary</u> to go to class to get a good grade, we can say "If you got a good grade, then you went to class."

On the other hand, if q is a <u>sufficient condition</u> for p to be true, then $q \to p$ (other way around!)
<u>E.g.</u> Since getting a B is <u>sufficient</u> to pass the class, we can say "If you got a B, then you passed the class."

So we see that it's important to treat $p \to q$ and $q \to p$ as different, but sometimes we want to assert <u>both</u>!

<u>Def'n</u> For prop.'s p and q, their <u>biconditional</u> is
$p \leftrightarrow q$: p <u>if</u> <u>and</u> <u>only</u> <u>if</u> q (same as $p \to q$ and $q \to p$)

Biconditional often used for <u>definitions</u>, and also for <u>logical</u> <u>equivalence</u>...

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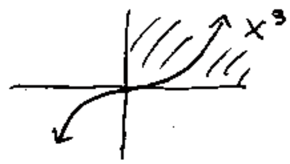
E.g. For any real number x, the biconditional "$x^3 > 0$ if and only if $x > 0$" is true because both:

- if $x > 0$, then $x^3 > 0$
- if $x^3 > 0$, then $x > 0$



E.g. Compare to: for any real number x, the conditional "If $x > 0$, then $x^2 > 0$" is true, but "If $x^2 > 0$, then $x > 0$" is false when $x = -1$.

The truth table for biconditional is

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \Leftrightarrow q$ is true if p and q have exactly same truth value (both true or both false)

Biconditionals let us define logical equivalence:

Def'n Suppose P and Q are compound propositions which depend on "input" propositions $p_1, p_2, \ldots, p_n$. Then we say that P and Q are logically equivalent, written $P \equiv Q$, if for all possible truth values of $p_1, p_2, \ldots, p_n$, P and Q have same truth value.

In other words, $P \Leftrightarrow Q$ for all $p_1, p_2, \ldots, p_n$

"P and Q are saying the same thing" if they are logically equivalent ...

E.g. Thm (De Morgan's Laws)
(1) $\neg(p \vee q) \equiv \neg p \wedge \neg q$ and (2) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Pf: Let's just verify the 1st De Morgan's Law.
The way we do this is by writing a truth table:

p	q	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

we see that they have same truth value no matter what, i.e., $(\neg(p \vee q)) \leftrightarrow (\neg p \wedge \neg q)$ ☒

E.g. Exercise Show that $p \equiv \neg(\neg p)$
(This is called "double negation.")

E.g. The contrapositive of the conditional $p \to q$ is $\boxed{\neg q \to \neg p}$. For instance, the contrapositive of "If $x > 0$, then $x^2 > 0$" is "If not$(x^2 > 0)$, then not$(x > 0)$," i.e., "If $x^2 \leq 0$, then $x \leq 0$."

Unlike the converse, the contrapositive is always logically equivalent to the original conditional!

Thm $p \to q \equiv \neg q \to \neg p$

Pf:

p	q	$p \to q$	$\neg q \to \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

↙ check the truth table!

☒