1/25 **Arguments and rules of inference** §1.4

Consider the following propositions:

- The murderer is Joe or Bob.
- The murderer is right-handed.
- Joe is not right-handed.

If these are all true, it is reasonable to conclude:

- Bob is the murderer.

Drawing a conclusion from a sequence of propositions like this is called *deductive reasoning*.

Def'n A sequence of propositions of the form: ⇒

$$\boxed{\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}}$$

is called a (*deductive*) *argument*.

The $p_1, \ldots, p_n$ are the *hypotheses* (or *premises*) and the q is the *conclusion*.

The "$\therefore$" symbol is read "therefore".

The argument is *valid* if:

whenever the hypotheses are all true, then the conclusion is also true!

(If it is not valid, we say it is *invalid*)

NOTE: Argument is valid $\neq$ argument is correct.

For example, the hypotheses could be false.

When we evaluate the validity of an argument, we analyze its form, not its content.

E.g. Thm

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

is a valid argument.

(This argument has a special name: it is called "*modus ponens*.")

Pf: One way to prove this is to write a truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

We see that whenever the both hypotheses $p \to q$ and p are true, then the conclusion q must also be true.

Can also just say by definition of $p \to q$, if $p \to q$ and p, then q. ∎

We give this argument the special name "modus ponens" because it is a basic rule of inference used often in the proofs of validity for other arguments.

Some other rules of inference are:

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$ "conjunction"

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$ "disjunction"

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$ "disjunctive syllogism"

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$ "hypothetical syllogism"

See §1.4 of book for more rules of inference...

Let's prove one more important one:

Thm
$$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$
is a valid argument. (It's called "modus tollens")

Pf: Since the contrapositive $\neg q \to \neg p$ is logically equivalent to $p \to q$, we can "replace" $p \to q$ w/ $\neg q \to \neg p$ to get an equivalent argument (valid if and only if original argument was valid).

But then $\neg q \to \neg p$, $\neg q / \therefore \neg p$ is an instance of modus ponens. ∎

See here the usefulness of logical equivalence for deductive reasoning...

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Now let's consider the 1^{st} argument we saw. Letting

p: The murderer is Joe.
q: The murderer is Bob.
r: The murderer is right-handed.

the argument has the form

$p \vee q$ ("Joe or Bob is murderer.")
r ("Murderer is right-handed.")
$p \to \neg r$ ("If Joe is murderer, then murderer is not right-handed.")
$\therefore q$ ("Therefore, murderer is Bob.")

This argument is *valid*, which we can prove as follows:

- We know r is equivalent to $\neg(\neg r)$ via "double negation"
- Then $\neg(\neg r)$ and $p \to \neg r$ yields $\neg p$ by modus tollens.
- Finally, $\neg p$ and $p \vee q$ yields q by disjunctive syllogism.

While it is theoretically always possible to use a truth table to prove the validity of an argument, using rules of inference is much more convenient...

Now let's look at an *invalid* argument:

If I get a B on the final, then I will pass the class.
I passed the class.
Therefore, I got a B on the final.

This argument has the form

$p \to q$ where p = "I get a B on the final"
q where q = "I pass the class"
$\therefore p$

It is *invalid* because $p \to q$ and q can both be true, while conclusion p is false!

e.g., maybe I got an A on the final

This kind of invalid argument is so common that it has a special name:

"the fallacy of affirming the consequent"

(there "fallacy" means "invalid argument.")

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Propositional formulas and Quantifiers §1.5

We mentioned earlier that basic math statements like

"n is an odd integer"

do not qualify as propositions because they involve a variable (like n) and may be true or false depending on the value of n. We will now consider these:

Def'n A propositional formula $P(x)$ is a statement involving a variable x, such that for each $x \in D$, $P(x)$ is a proposition (i.e., either true or false). Here D is a set called the domain of discourse.

E.g. If the domain of discourse is the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of nonnegative integers, then $P(n) =$ "n is an odd integer" is a propositional formula.

For each $n \in \mathbb{N}$, it determines a proposition:

$P(1) =$ "1 is an odd integer," which is true

$P(2) =$ "2 is an odd integer," which is false

Knowing the domain of discourse D of a prop. formula is very important, but D is often implicit.

E.g. $P(x) =$ "$x^2 \geq 0$" is a prop. formula, where we implicitly assume domain of discourse is set of real numbers $\mathbb{R}$.

Note: often use n for integer, x for real number.

Something is special about this $P(x)$: for every real number $x \in \mathbb{R}$, prop. $P(x) =$ "$x^2 \geq 0$" is true.

We will often want to talk about claims like this:

Def'n If $P(x)$ is a prop. formula w/ domain of discourse D, the statement "for every $x \in D$, $P(x)$" (often abbreviated "for every x, $P(x)$") is called a universally quantified statement. It is denoted symbolically as

$$\forall x \ P(x)$$

where the symbol "$\forall$" is read "for all."

Even though $P(x)$ by itself is not a proposition, $\forall x \ P(x)$ is a proposition, and it is true exactly when for all $x \in D$, $P(x)$ is true.

E.g. The proposition "$\forall x, x^2 \geq 0$" is true (where we assume domain of discourse is $D = \mathbb{R}$): this expresses the well-known property of real numbers, that their squares are nonnegative.

E.g. The proposition "$\forall x, x^2 > 0$" is false (again assuming $D = \mathbb{R}$) since for $x = 0$ we have that $x^2 = 0^2 = 0$, which is not > 0.

strict inequality

Notice: to show a universally quantified statement is false, just have to exhibit one counterexample. A counterexample is a $x \in D$ s.t. $P(x)$ is false. On the other hand, to show $\forall x \ P(x)$ is true, have to prove $P(x)$ is true for every $x \in D$.

E.g. The statement "Every planet in the solar system has a moon" is a universally quantified statement;

- discourse domain $D = \{\text{planets in solar system}\}$
- prop. formula is $P(x) =$ "x has a moon."

It is false, since Mercury has no moons (nor does Venus).

E.g. Consider a different kind of statement:
"There is some planet in the solar system which has a moon."

This proposition is true: Earth has a moon (as do other planets...)

This is called an existentially quantified statement:

Def'n For prop. formula $P(x)$ w/ discourse domain D, the statement "there is an $x \in D$ such that $P(x)$" (or "there exists x s.t. $P(x)$") is an existentially quantified statement. It is written symbolically as

$$\exists x \; P(x), \quad \text{where } \exists = \text{"there exists"}$$

The proposition $\exists x \; P(x)$ is true exactly when there is at least one $x \in D$ such that $P(x)$ is true.

E.g. The statement "$\exists x, x^2 = 9$" is true (assuming $D = \mathbb{R}$) since for $x = 3$ we have $x^2 = 3^2 = 9$ (and also for $x = -3$).

Just need to find one x s.t. $P(x)$ is true!

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You might think that "for all" and "there exists" statements seem "opposite" to each other, in same way that and & or are "opposite." This is true:

Thm (Generalized De Morgan's Laws)

(1) $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$

(2) $\neg(\exists x\ P(x)) \equiv \forall x\ \neg P(x)$

Pf: We prove only (1) since (2) is very similar.
$\neg(\forall x\ P(x))$ means exactly that there is some $x \in D$ for which $P(x)$ is false, i.e., for which $\neg P(x)$ is true. But this is exactly what $\exists x\ \neg P(x)$ means too. $\boxtimes$

Related to usual De Morgan's Laws because if $D = \{x_1, x_2, \dots, x_n\}$ then

$\neg(\forall x\ P(x))$ means $\neg(P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n))$

while $\exists x\ \neg P(x)$ means $(\neg P(x_1) \vee \neg P(x_2) \vee \cdots \vee \neg P(x_n))$

which are logically equiv. by De Morgan for $\wedge$ & $\vee$.

Eg. Let $P(x) = $ "$\frac{1}{x^2+1} > 1$" (w/ $D = \mathbb{R}$ as usual).

We can prove $\exists x\ P(x)$ is false by showing instead that $\forall x\ \neg P(x)$ is true, as follows:

Recall that $\forall x \in \mathbb{R},\ x^2 \geq 0$,

so that $\forall x \in \mathbb{R},\ x^2 + 1 \geq 1$

Dividing both sides by (x^2+1) (which is ≥ 1) gives

$$\forall x \in \mathbb{R},\ 1 \geq \frac{1}{x^2+1}$$

which is the same as

$$\forall x \in \mathbb{R},\ \neg\left(\frac{1}{x^2+1} > 1\right),$$

i.e., $\forall x \in \mathbb{R},\ \neg P(x)$. //

Warning: Translating quantified English statements to their symbolic logic versions can be even more tricky ... have to use common sense!

E.g. Consider the famous idiom:

(*) "All that glitters is not gold."

(This just means "not everything is what it seems.")

If we let $P(x)$ = "x glitters"

and $Q(x)$ = "x is gold"

then a hyper-literal translation of (*) would be

$$\forall x, (P(x) \to \neg Q(x)),$$

i.e., "for every thing, if that thing glitters, then it is not gold."

But the real meaning of (*) is instead:

$$\neg(\forall x \; P(x) \to Q(x)),$$

i.e., "It is not the case that everything that glitters is gold."

Upshot: English is not very consistent about where to put negatives in universally quantified statements.

Exercise: Take other common idioms like

"Not all those who wander are lost",

"Everyone has their price", etc.

and convert them to symbolic logic statements.