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Relations §3.3

You can think of a relation from one set X to another set Y as a chart that records how elements from X are "related" to elements from Y. For example, we can consider a chart that records for each student in a school the classes they're taking:

Student	Class
Bill	Economics
Bill	English
Alexis	English
Jordan	Chemistry ...

Notice that unlike a function, each student can take multiple classes. Also, a student may be taking no classes at all (e.g. they're on a leave of absence).

Def'n Formally, a (binary) relation R from set X to set Y is any subset of $X \times Y$, i.e., any set of ordered pairs of form (x, y) with $x \in X$ and $y \in Y$. If $(x,y) \in R$ then we write $x R y$ and we say "x is related to y."

E.g: For the student/class example, the relation is

$$R = \{(\text{Bill}, \text{Econ.}), (\text{Bill}, \text{Eng.}), (\text{Alexis}, \text{Eng.}), (\text{Jordan}, \text{Chem.}), \ldots\}$$

and since Alexis is taking English we could also write Alexis R English.

Notice: A function $f: X \to Y$ is a very special relation from X to Y: one for which each $x \in X$ is related to exactly one $y \in Y$.

But relations can model things that functions can't...

The most important relations are when $X = Y$:

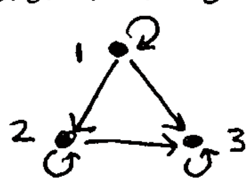
<u>Def'n</u> If R is a relation from X to X, we say it is a <u>relation on the set X</u>.

<u>E.g.</u> If $X = \{1,2,3\}$ then $\leq$ defines a relation on X: we have "a is related to b" if and only if "$a \leq b$".

The set of ordered pairs for this relation is:

$$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$$

We can represent this same information with a <u>digraph</u>:



Here we draw a "<u>vertex</u>" (a dot •) for each element of X, and draw an <u>arrow</u> $a \bullet\!\!\longrightarrow \bullet b$ whenever $a \, R \, b$.

Notice that if $a R a$ then we have a <u>loop</u>: $\underset{a}{\bullet}\circlearrowleft$

<u>Def'n</u> The relation R is called <u>reflexive</u> if $x R x$ for all $x \in X$.

<u>E.g.</u> The $\leq$ relation on $\{1,2,3\}$ is reflexive: means we have a <u>loop</u> at every vertex.

But if we consider the $<$ relation instead:

this is <u>not</u> reflexive (no loops at all here).

Reflexivity captures the difference between $\leq$ (less than or equal to) and $<$ (strictly less than).

Def'n The relation R is called symmetric if whenever $x R y$ then also $y R x$, for all $x, y \in X$.

E.g. The relation $\leq$ on $\{1,2,3\}$ is not symmetric, since $1 \leq 2$ but $2 \not\leq 1$.

For a symmetric relation the digraph looks like:

$a \bullet \rightleftarrows \bullet b$ or $a \bullet \quad \bullet b$ (no arrows) for every a, b.

E.g. An example of a symmetric relation R is $X = \{\text{Students at Howard}\}$ and $x R y$ means "x has a class with y."

This is symmetric since if Person x has a class with Person y, then Person y has class with Person x!

Relations $\leq$ are "opposite" from symmetric, so:

Def'n Relation R is called anti-symmetric if whenever $x R y$ and $y R x$ then $x = y$, for all $x, y \in X$.

E.g. The relation $\leq$ (on $X = \{1,2,3\}$ or $X = \mathbb{Z}$ or $X = \mathbb{R}, \ldots$) is anti-symmetric since if $x \leq y$ and $y \leq x$ then we must have $x = y$.

The relation $<$ is also anti-symmetric: there are no x, y at all with $x < y$ and $y < x$.

For an anti-symmetric relation digraph is:

No $\cancel{a \bullet \rightleftarrows \bullet b}$ but loops $\overset{\circlearrowleft}{\underset{a}{\bullet}}$ OK ✓

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There is one more important property of $\leq$:

<u>Def'n</u> A relation R on X is called <u>transitive</u> if for all $x, y, z \in X$, wherever we have $x R y$ and $y R z$ then we must have $x R z$:

$\Leftarrow$ must have the dotted arrow

<u>E.g.</u> The relation $\leq$ (or $<$) is transitive because if $a \leq b$ and $b \leq c$ then certainly $a \leq c$.

<u>Q</u>: Is relation "has a class with" on students transitive?

<u>A</u>: <u>No</u>! Maybe Bill has English class with Alexis, and Alexis has Biology class with Cole, but Bill has no class with Cole.

<u>Def'n</u> A relation R on X that is:
- <u>reflexive</u>,
- <u>anti-symmetric</u>,
- and <u>transitive</u>,

is called a <u>partial order</u> on X.

<u>E.g.</u> $\leq$ is a partial order on $X = \{1,2,3\}$ (or on X = any set of numbers).

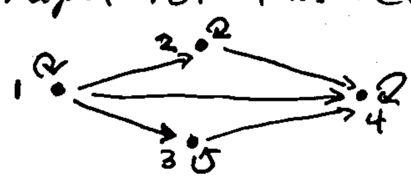
Partial orders behave like $\leq$: they let us "<u>compare</u>" things in X.

But... partial orders don't necessarily let us compare <u>every</u> pair of elements...

E.g.: Consider a list of tasks you must do to complete a project.
Maybe the project is "make a PB & J Sandwich"
and the tasks are:

1. Toast two slices of bread.
2. Spread peanut butter on one slice.
3. Spread jelly on the other slice.
4. Put the two slices together.

Some of the tasks must be done before others (1 must be done before 2)
So we can define a relation R on the set of tasks:
$i \, R \, j$ if $i = j$ or task i must be done before task j
The digraph for this relation is:



reflexive ✓
anti-symmetric ✓
transitive ✓

There are no arrows between 2 and 3 since these two tasks can be done in either order.
Notice: we get a partial order on the tasks!

If R is a partial order on X and $x, y \in X$,
we say x and y are comparable if $x R y$ or $y R x$,
and say they are incomparable otherwise.

E.g. In PB&J example, tasks 2 and 3 are incomparable (can be done in either order).

The partial order R on X is called a total order if every pair $x, y \in X$ is comparable.

E.g. Relation $\leq$ on any set of #'s is a total order, but the "do before" relation on tasks is not a total order!

Compositions of relations and inverse relations §3.3

Now we return to discussing relations R from X to Y. Recall that a function $f: X \to Y$ is a special such relation, and we can generalize to relations the important functional notions of composition and inversion.

Def'n Let R_1 be a relation from X to Y, and R_2 a relation from Y to Z. The composition $R_2 \circ R_1$ is a relation from X to Z where for $x \in X$ and $z \in Z$ we have $x\,(R_2 \circ R_1)\,z$ if and only if there is $y \in Y$ with $x \, R_1 \, y$ and $y \, R_2 \, z$.

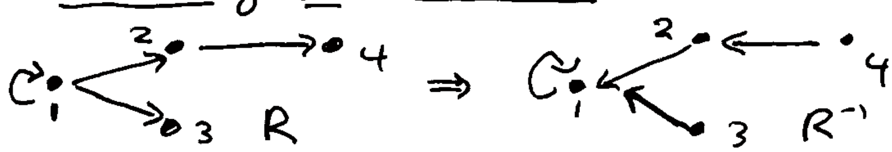
$$X \xrightarrow{R_1} Y \xrightarrow{R_2} Z \implies X \xrightarrow{R_2 \circ R_1} Z.$$

Def'n Let R be a relation from X to Y. The inverse relation R^{-1} is a relation from Y to X where $R^{-1} = \{(y, x) : (x, y) \in R\}$

"reverse" every ordered pair in R

Note: For function $f: X \to Y$, the inverse $f^{-1}: Y \to X$ is defined only when f is a bijection. But the inverse relation R^{-1} is always defined.

– If R is a relation on X (i.e. from X to X) then the digraph of R^{-1} is obtained from digraph of R by reversing the direction of all arrows.



Q: What is inverse of $\leq$? **A**: $\geq$!

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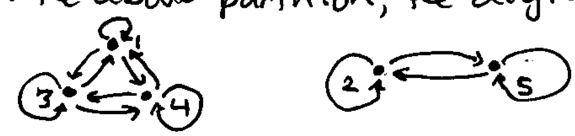
Equivalence Relations §3.4

Let X be a set and recall that a partition of X is a collection S of (nonempty) subsets of X such that every $x \in X$ belongs to exactly one subset in S.

E.g. For $X = \{1,2,3,4,5\}$ one partition is $S = \{\underbrace{\{1,3,4\}}, \underbrace{\{2,5\}}\}$.

A partition S is a way of "breaking X into groups" and we can use S to define a relation R on X where $x R y$ if and only if x and y are in same subset in S.

E.g. with the above partition, the digraph of R is:



Theorem Relation R defined from a partition S of X is:
- reflexive
- symmetric
- and transitive.

Pf: All properties are easy to check directly.
Reflexive: x is in the same subset of S as itself.
Symmetric: if x is in same subset as y, then vice-versa.
Trans.: if x is same subset as y, and y as z, then same for x and z. ☒

Def'n Any relation R on a set X that is:
- reflexive
- symmetric ← (compare to def. of partial order)
- and transitive

is called an equivalence relation on X.

An equivalence relation on X is a way that elements of X can be "the same".

E.g. Relation R on $\mathbb{R}$ where xRy if $x^2=y^2$ is an equiv. relation.
E.g. Let n be any positive integer. We define relation R on $\mathbb{Z}$ where xRy if $x-y$ is a multiple of n.
Exercise: This is an equivalence relation on $\mathbb{Z}$.

Partitions give us equivalence relations, and conversely:

Thm Let R be an equiv. relation on X. Let $a \in X$ be any element and define $[a] := \{x \in X : x R a\}$ (things related to element a).
Then $S = \{[a] : a \in X\}$ is a partition of X.

Pf: Need to show every $x \in X$ belongs to exactly one subset in S.
By reflexivity of R, have $x \in [x]$. So suppose $x \in [y]$.
Want to show then that $[x]=[y]$. So let $z \in [x]$.
Then zRx, and since xRy, have zRy by transitivity, i.e., have $z \in [y]$. By symmetry have yRx, so for any $z \in [y]$ have $z \in [x]$ by same argument. Thus, $[x]=[y]$. ∎

Def'n The sets $[a]$ for $a \in X$ from the previous theorem are called the equivalence classes of the equiv. relation R.

E.g. With R being equiv. relation on $\mathbb{R}$ where xRy if $x^2=y^2$, equivalence classes are $\{a,-a\}$ for $a \in \mathbb{R}$, i.e., each number is grouped with its negative.

E.g. Exercise What are the equivalence classes for the "xRy if $x-y$ is a multiple of n" equivalence relation on the integers $\mathbb{Z}$?
Hint: Consider modular arithmetic mod n.

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Combinatorics: Basic Counting Principles §6.1

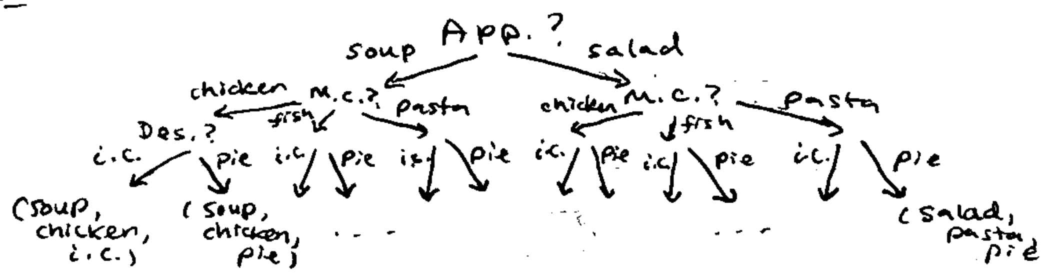
We are starting a new chapter (our last of the semester): Chapter 6, on combinatorics, a fancy word for "counting." We'll learn many techniques for counting finite collections. We start with some basic counting principles:

E.g.: Suppose that for a meal you must choose:

- an appetizer: either soup or salad,
- a main course: chicken, fish, or pasta,
- a dessert: either ice cream or pie.

Q: How many different meals is it possible to make?

A: We can represent all the choices in a "decision tree":



We see that there are $12 = 2 \times 3 \times 2$ total meals.

We multiply the choices at each step to get total!

Thm (Multiplication Principle for Counting)

Suppose we make an object via a series of steps, where we have k_1 choices for step 1, k_2 choices for step 2, down to k_m choices for step m. Then the total # of objects we can make is $k_1 \times k_2 \times \cdots \times k_m$.

Remark: We saw before that for product $X_1 \times X_2 \times \cdots \times X_m$ of sets $X_1, X_2, \ldots, X_m$ which are finite, we have

$$\#(X_1 \times X_2 \times \cdots \times X_m) = \#X_1 \times \#X_2 \times \cdots \times \#X_m.$$

This is basically the same as the multiplication principle.

Let's see some more examples of the multiplication principle:

E.g. A US telephone # has 10 digits, & first digit cannot be 0.

Q: How many telephone #'s are there?

A: We have 9 possibilities for the 1st digit, and 10 for each of the 9 others. So by mult. principle:

$9 \times \overbrace{10 \times 10 \times \cdots \times 10}^{9 \text{ times}} = 9 \times 10^9 = 9$ billion telephone numbers.

E.g. We saw before that the # of subsets of set $\{1, 2, 3, \ldots, n\}$ is 2^n.

To make a subset, we decide:
- Include 1 or not? (2 choices)
- Include 2 or not? (2 choices)
- $\vdots$
- ... Include n or not? (2 choices)

This is n ~~choice~~ steps, with 2 choices at each step, so by mult. principle, # possibilities $= \overbrace{2 \times 2 \times \cdots \times 2}^{n \text{ times}} = 2^n$.

E.g. Q: How many relations on $X = \{1, 2, \ldots, n\}$ are there?

A: For each pair $(x, y) \in X \times X$, we can choose to include (x, y) in our relation R on X, or not. There are $\#X \cdot \#X = n^2$ total pairs of the form (x, y), so we build a relation in n^2 steps, with 2 choices at each step.

This gives $\overbrace{2 \times \cdots \times 2}^{n^2 \text{ times}} = 2^{n^2}$ possibilities.

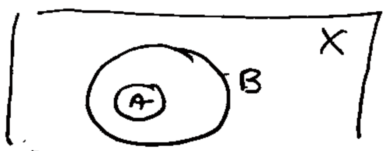
Can also just say that a relation R is any subset of $X \times X$, a set of size n^2, so again get 2^{n^2} such subsets.

Exercise: How many symmetric relations on $X = \{1, 2, \ldots, n\}$ are there? What about reflexive?

E.g. Let $X = \{1, 2, \ldots, n\}$ as before.

Q: How many ordered pairs (A, B) of subsets of X satisfying $A \subseteq B \subseteq X$ are there?

A: It is helpful to draw a Venn diagram of our situation:



$\Leftarrow$ We see that the Venn diagram has 3 regions:

- things in A,
- things in $B \setminus A$,
- things in $X \setminus B$.

So to make an ordered pair (A, B) of desired form, we can choose for each $i = 1, 2, \ldots, n$ which of the three regions to place i into:

- Put 1 in A, $B \setminus A$, or $X \setminus B$? (3 choices)
- Put 2 in A, $B \setminus A$, or $X \setminus B$? (3 choices)

$\vdots$

- Put n in A, $B \setminus A$, or $X \setminus B$? (3 choices)

Thus, we have n steps with 3 choices at each step, so total # of possibilities $= 3 \times 3 \times \cdots \times 3 = \boxed{3^n}$

Exercise What about (A, B, C) with $A \subseteq B \subseteq C \subseteq \{1, 2, \ldots, n\}$?

And (A, B, C, D)? And so on...?

4/3 Addition Principle + Principle of Inclusion-Exclusion

Sometimes we are trying to count objects that have multiple "kinds":

E.g. Q: Let $X = \{a,b\}$. How many strings in X^* are there which have length 3 or length 4?

A: The # of strings of length 3 in $X^* = 2 \times 2 \times 2 = 2^3$ by mult. principle
# of strings of length 4 $= 2 \times 2 \times 2 \times 2 = 2^4$
# of strings of length 3 or 4 $= 2^3 + 2^4 = 8 + 16 = \underline{24}$.

We see another counting principle in action here:

Theorem (Addition Principle for Counting)
If $X_1, X_2, \dots, X_m$ are disjoint sets (meaning $X_i \cap X_j = \emptyset$ for all $i \neq j$, i.e., the sets have no common elements) then $\#(X_1 \cup X_2 \cup \cdots \cup X_m) = \#X_1 + \#X_2 + \cdots + \#X_m$.

We see that, as long as the sets are disjoint, we can count any grouping of sets just by adding together:

E.g. Q: # of strings in $\{a,b\}^*$ of length 3 or 4 or 5?
A: $2^3 + 2^4 + 2^5$, by the addition principle.

E.g. Alexis, Ben, Cole, David and Erica are a 5 person group. They have to elect a: President, Vice President, & Treasurer.

Q: a) How many ways are there to do this?
b) How many ways are there if we require that either Alexis or Ben is the President?

A: a) We can choose any of the 5 people for Prez.
Then for VP we can choose any of the remaining 4.
And for treas. we can choose any of remaining 3.
By the mult. principle this gives: $5 \times 4 \times 3 = \underline{60}$ ways...

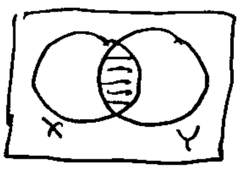
b) If Alexis is Prez., we have $4 \times 3 = 12$ ways to choose VP + Treas.
If Ben is Prez., also have $4 \times 3 = 12$ ways to choose VP + Treas.
By addition principle, the total # of ways $= 12 + 12 = \underline{\underline{24}}$.

But what if the sets are <u>not</u> disjoint? Then we use:

<u>Theorem</u> (<u>Principle</u> <u>of</u> <u>Inclusion</u> - <u>Exclusion</u>)

$$\#(X \cup Y) = \#X + \#Y - \#(X \cap Y)$$ ← notice that if X and Y are disjoint then this term is $\underline{\underline{0}}$.

To see why P.I.E. works, look at a Venn diagram:



← when we add $\#X$ and $\#Y$ we count things in $X \cap Y$ <u>double</u>, so have to subtract $-\#(X \cap Y)$ to correct.

<u>E.g.</u> <u>Q</u>: c) How many ways to pick Prez., VP, & treasurer where <u>either</u> Alexis is <u>Prez.</u> or Ben is <u>VP</u> (or both)?

<u>A</u>: c) Let X = elections where Alexis is Prez.
Then $\#X = 4 \times 3 = 12$, # of choices of VP + treas.
Let Y = elections where Ben is VP
Then $\#Y = 4 \times 3 = 12$, # of choices of Prez. + Treas.
We want to compute $\underline{\#(X \cup Y)}$.
By P.I.E., we also need to know $\#(X \cap Y)$:
$\#(X \cap Y) = 3$, since if Alexis is Prez. & Ben is VP, there are 3 choices left for Treas.

So... $\#(X \cup Y) = \#X + \#Y - \#(X \cap Y) = 12 + 12 - 3$
$= \underline{\underline{21}}$ ways for Alexis to be Prez. <u>or</u> Ben to be VP.