4/5 <u>Permutations</u> <u>and</u> <u>Combinations</u> §6.2

<u>Def'n</u> A <u>permutation</u> of n distinct elements $x_1, x_2, \ldots, x_n$ is an ordering of the elements, i.e., a <u>list</u> of the elements where each x_i appears e<u>xactly</u> <u>once</u>.

<u>E.g.</u> There are 6 permutations of A, B, C:

ABC ACB BAC BCA CAB CBA

Recall that for a positive integer $n \geq 1$, we defined <u>n factorial</u> as $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

Also makes sense to define $\boxed{0! = 1}$

<u>Theorem</u> The # of permutations of n elements is $n!$

<u>Pf</u>: Create a permutation by choosing 1^{st} element in list, then $2^{nd}, \ldots$, down to n^{th}. There are n choices for 1^{st}. Then there are $(n-1)$ choices for 2^{nd} (since 1^{st} is not available) $(n-2)$ choices for 3^{rd}, etc., down to 1 choice for n^{th}. By mult. principle, gives $n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$ total. ☒

We can also do a slightly more general thing:

<u>Def'n</u> An <u>r-permutation</u> of $x_1, \ldots, x_n$ is a length r list of elements in $x_1, \ldots, x_n$ where each appears <u>at most once</u>. (We need $r \leq n$ for such a list to exist.)

<u>E.g.</u> There are 12 2-permutations of A, B, C, D:

AB AC AD BA BC BD CA CB CD DA DB DC.

We use $P(n,r) :=$ # of r-permutations of n elt. set.

<u>Thm</u> $P(n,r) = n \times (n-1) \times \cdots \times (n-r+1) = \dfrac{n!}{(n-r)!}$

<u>Pf</u>: Same as proof for usual permutations, but stop after the r^{th} step. ☒

We often want to count <u>unordered</u> collections of given size.

<u>Def'n</u> An <u>r-combination</u> of $x_1, \dots, x_n$ is a length r <u>unordered</u> collection of elements in $x_1, \dots, x_n$, i.e., a size r <u>subset</u> of $\{x_1, \dots, x_n\}$.

<u>E.g.</u> There are 6 2-combinations of A, B, C, D:

$\{A,B\}$ $\{A,C\}$ $\{A,D\}$ $\{B,C\}$ $\{B,D\}$ $\{C,D\}$

Let $C(n,r) = \#$ r-combinations of n element set

(common notation $\binom{n}{r}$ - read 'n choose r' - used too...)

How to give a formula for $C(n,r)$?

We can create an <u>r-permutation</u> of $x_1, \dots, x_n$ as follows:

1. Pick one of the $C(n,r)$ r-combinations, call it $\{y_1, \dots, y_r\} \subseteq \{x_1, \dots, x_n\}$
2. Choose one of the $r!$ permutations of $y_1, \dots, y_r$.

<u>E.g.</u> To make a 2-permutation of A, B, C, D, we first pick one of the 6 2-combinations, and then choose one of the $2! = 2$ ways to permute its letters:

$$\begin{array}{cccccc} \{A,B\} & \{A,C\} & \{A,D\} & \{B,C\} & \{B,D\} & \{C,D\} \\ \swarrow \ \searrow & \swarrow \ \searrow & \swarrow \ \searrow & \swarrow \ \searrow & \swarrow \ \searrow & \swarrow \ \searrow \\ AB \quad BA & AC \quad CA & AD \quad DA & BC \quad CB & BD \quad DB & CD \quad DC \end{array}$$

By the multiplication principle, this means

$$\begin{array}{c}\text{\# of ways to make} \\ \text{r-permutation} \\ \text{of } x_1, \dots, x_n\end{array} = \begin{array}{c}\text{\# ways to make} \\ \text{r-combination} \\ \text{of } x_1, \dots, x_n\end{array} \times \begin{array}{c}\text{\# of permutations} \\ \text{of r things}\end{array},$$

i.e. $P(n,r) = C(n,r) \times r!$

$\Rightarrow$ <u>Theorem</u> $C(n,r) = \dfrac{P(n,r)}{r!} = \dfrac{n!}{(n-r)!\,r!}$

4/7

E.g. We saw there were 6 2-combinations of A, B, C, D, and $C(4,2) = \frac{4!}{2!2!} = \frac{4\times3\times2\times1}{2\times1\times2\times1} = 6.$ ✓

We will have a lot more to say about these $C(n,r)$ in a little bit. Here is just a taste:

Exercise Show $\sum_{r=0}^{n} C(n,r) = 2^n$.

Hint: Imagine choosing an arbitrary subset of $\{1,2,\dots,n\}$ by first choosing its size r.

E.g. A standard deck of cards has 52 cards in it:

- there are 4 suits: spades ♠, hearts ♡, clubs ♣, diamonds ◇
- there are 13 ranks: 2–10 and J Q K A

for a total of $4\times 13 = 52$ different cards.

A poker hand consists of 5 of these 52 cards.

Q: 1) How many poker hands are there?

2) How many poker hands have cards of all the same suit (this is called a "flush")?

A: 1) Since a poker hand is an unordered subset of size 5 from 52 elements, there are $C(52,5) = 2{,}598{,}960$ different poker hands

2) To make a flush, first pick the suit of all the cards, then select 5 of the 13 ranks for the hand

$\Rightarrow 4\times C(13,5) = 4\times 1287 = 5{,}148$ flushes.

This means $\approx 0.2\%$ of hands are flushes (very rare!) //

4/14 <u>Generalized</u> <u>Permutations</u> §6.3

There are $n!$ permutations of n <u>distinct</u> letters:

ABC ACB BAC BCA CAB CBA

But what about rearrangements of a word with <u>repeated</u> letters?

<u>E.g.</u> How many ways are there to rearrange the letters in

MISSISSIPPI ?

Some of the 11! permutations will "be the same", so the answer is something less than 11!.

Let's start with something easier: how to count rearrangements of AAABBBBB.

A rearrangement is 8 letters, 3 of them A's, 5 B's:

_ _ _ _ _ _ _ _

Of the 8 positions for letters, we can select any 3 for A's, and then the 5 B's must go in the other positions:

B B A B A B A B

We are choosing 3 spots out of 8, which gives

$C(8,3) = 8!/(3!\cdot 5!) = 56$ total rearrangements.

For MISSISSIPPI, we can do similarly, but in more steps.

We have 11 spots, choose 4 of them for the I's:

_ I _ I I _ I _ _ _ _ $C(11,4)$

Then from remaining 7 spots, choose 4 for the S's:

_ I S I S _ I S S _ S $C(7,4)$

Then from remaining 3 spots, choose 2 for the P's:

P I S I I _ I S S P S $C(3,2)$

The M goes in remaining spot in $C(1,1)$ ways.

Altogether, there are $C(11,4)\cdot C(7,4)\cdot C(3,2)\cdot C(1,1)$

$$= \frac{11!}{4!\cancel{7!}}\cdot\frac{\cancel{7!}}{4!\cancel{3!}}\cdot\frac{\cancel{3!}}{2!\cancel{1!}}\cdot\frac{\cancel{1!}}{1!\cancel{0!}} = \frac{11!}{4!\,4!\,2!\,1!}$$

$= 34{,}650$ rearrangements of MISSISSIPPI.

Theorem For a word which has m different kinds of letters, with n_1 of the 1^{st} letter, n_2 of the 2^{nd} letter, ... and n_m of the m^{th} letter, so $n = n_1 + n_2 + \cdots + n_m$ total letters, the # of rearrangements $= n!/(n_1! \cdot n_2! \cdot n_3! \cdots n_m!)$.

Pf: Same as what we just explained! $\boxtimes$

E.g. MISSISSIPPI $\Rightarrow n = 11$, $n_1 = 4$ I's, $n_2 = 4$ P's, $n_3 = 2$ S's, $n_4 = 1$ M

So that # rearrangements $= 11!/(4!\,4!\,2!\,1!)$.

Notice that if all letters are distinct, so $n_1 = n_2 = \cdots = n_m = 1$, we get usual $n!/(1! \cdot 1! \cdots 1!) = n!$ permutations, and the more repeated letters we have, the more we have to divide $n!$ by to account for the repeat.

In fact, another way to think about the formula: if we put subscripts (or 'colors') on repeated letters, like:

$$M_1\, I_1\, S_1\, S_2\, I_2\, S_3\, S_4\, I_3\, P_1\, P_2\, I_4$$

then all these letters become 'distinct', so that there are $n!$ $(=11!)$ different permutations of the subscripted letters. And then... Given any rearrangement (without subscripts), there are $n_1!\, n_2! \cdots n_m!$ $(= 4!\,4!\,2!\,1!)$ ways to put subscripts on all the repeated letters.

So dividing $n!$ by $n_1!\, n_2! \cdots n_m!$ gives us the number of ways of rearranging the letters (similar to how dividing $P(n,r)$ by $r!$ gave $C(n,r)$). //

4/17 <u>Generalized</u> <u>combinations</u> §6.3

Last class we saw how to deal with repeats in permutations. What about <u>combinations</u> where we allow repeats?

<u>E.g.</u>: At a bagel shop they have four flavors of bagels: plain, sesame, everything, & cinnamon raisin. You want to buy 13 bagels (= a baker's dozen). How many ways are there to do this?

If we had to pick 13 <u>distinct</u> flavors of bagels, this would be a $C(n,k)$ combinations problem. But of course we can <u>repeat</u> flavors in our purchase.

There is a very nice <u>trick</u> for these kinds of problems called "<u>stars and bars</u>", where we represent a bagel purchase by a picture that looks like this:

$* * *$	$\vert$	$* *$	$\vert$	$* * * * *$	$\vert$	$* * *$
plain		sesame		everything		cinnamon raisin

This means that we buy 3 plain, 2 sesame, 5 everything, and 3 cinnamon raisin bagels.

Any pattern of 13 $*$'s ('stars') and 3 $\vert$'s ('bars') gives us a bagel purchase: the $*$'s represent the bagels, with the $\vert$'s serving as 'dividers' between bins representing the 4 flavors.

So to count bagel purchases, we just need to count patterns of 13 $*$'s and 3 $\vert$'s. But this is exactly the <u>word</u> <u>rearrangement</u> problem, where we saw the answer is:

$C(16,13) = 16!/13!\,3! = \underline{560}$.

In general, we have the following formula for counting combinations with repeats allowed:

<u>Theorem</u> The number of ways to select k things from m options, allowing selecting an option multiple times is $C(k+m-1,k) = C(k+m-1, m-1) = \frac{(k+m-1)!}{(m-1)!\,k!}$.

(Notice that we always have $C(n,k) = C(n, n-k)$.)

E.g.: You have 11 identical candies to give to 3 children. How many different ways can you distribute the candies?

<u>Idea</u>: represent a candy distribution like:

$$\underbrace{* * * *}_{\substack{\text{4 candies} \\ \text{1st child}}} \mid \underbrace{* *}_{\substack{\text{2 candies} \\ \text{2nd child}}} \mid \underbrace{* * * * *}_{\substack{\text{5 candies} \\ \text{3rd child}}}$$

This 'stars and bars' trick shows it is the same as the bagel problem, and so there are $C(11+2, 11) = 13!/11!\cdot 2! = \underline{78}$ ways to give candies.

<u>Q</u>: What if we are required to give each child at least one candy?

<u>A</u>: First give each child one candy. This leaves $(11-3) = 8$ candies which can be distributed arbitrarily in $C(8+2, 8) = 10!/8!\cdot 2! = \underline{45}$ ways.

//

§6.7

4/19 Binomial coefficients and the Binomial Theorem

We start with an algebra exercise:

$$\begin{aligned}(a+b)^n &= (a+b)(a+b)(a+b)\\ &= aaa + aab + aba + abb\\ &\quad + baa + bab + bba + bbb\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

where a and b can be any numbers (or variables)

What's the significance of this sequence 1, 3, 3, 1?

If we expanded:

$$(a+b)^4 = \cdots = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

we'd get the coefficient sequence 1, 4, 6, 4, 1.

And in general we have...

Theorem (Binomial Theorem)

$$(a+b)^n = \sum_{k=0}^{n} C(n,k)\, a^{n-k} b^k.$$

Pf: Imagine expanding $(a+b)^n$:

$(a+b)(a+b) \cdots (a+b)$ ← n terms in total

If we want to make a term of $a^{n-k}b^k$ from these multiplications, we have to choose the "b" part from exactly k of the $(a+b)$'s and the "a" part from the n−k other $(a+b)$'s.

Thus, the number of ways to do this is the # of ways to choose k positions from n, which by definition is $C(n,k) = \frac{n!}{k!(n-k)!}$. ▨

Note: In this context, also use notation $\binom{n}{k} = C(n,k)$ for the "n choose k" numbers: $\sum_{n=0}^{n} \binom{n}{k} a^{n-k} b^k = (a+b)^n$

The $\binom{n}{k}$ are also called binomial coefficients.

Using the binomial theorem, we can give short proofs of some identities we've already seen, like:

<u>Theorem</u> $\sum_{k=0}^{n} C(n,k) = 2^n$

<u>Pf</u>: Bin. Thm. says $\sum_{k=0}^{n} C(n,k) = (a+b)^n$

Let $a=1$ and $b=1$ $\Rightarrow \sum_{k=0}^{n} C(n,k)\, 1^{n-k} 1^k = (1+1)^n$

$$\Rightarrow \sum_{k=0}^{n} C(n,k) = 2^n. \quad ☒$$

What about the <u>alternating sum</u> of the $C(n,k)$'s?

<u>E.g.</u> $C(3,0) - C(3,1) + C(3,2) - C(3,3)$
$= 1 - 3 + 3 - 1 = \underline{0}$

or $C(4,0) - C(4,1) + C(4,2) - C(4,3) + C(4,4)$
$= 1 - 4 + 6 - 4 + 1 = \underline{0}$.

<u>Theorem</u> For $n \geq 1$, $\sum_{k=0}^{n} (-1)^k C(n,k) = 0$.

<u>Proof</u>: Let $b=-1$ and $a=1$ in the Binomial Theorem:

$$\sum_{k=0}^{n} (-1)^k C(n,k) = (1-1)^n = 0^n = 0. \quad ☒$$

<u>NOTE</u>: $C(0,0) = \frac{0!}{0!\,0!} = 1$, so for $n=0$ we have

$$\sum_{k=0}^{n} (-1)^k C(n,k) = C(0,0) = 1,$$

which means we should interpret 0^0 as $\underline{1}$.

The HW has other identities for the $C(n,k)$ which can be proved using the Binomial Theorem...

4/21 Pascal's Triangle §6.7

The Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} C(n,k)\, x^k y^{n-k}$ suggests that we should look at the sequence $C(n,0), C(n,1), C(n,2), \ldots, C(n,n)$ in a "row"

Actually, we can put all these rows together into an infinite triangular array:

$$
\begin{array}{ccccccc}
 & & & C(0,0) & & & \\
 & & C(1,0) & & C(1,1) & & \\
 & C(2,0) & & C(2,1) & & C(2,2) & \\
C(3,0) & & C(3,1) & & C(3,2) & & C(3,3) \\
 & & & \vdots & & &
\end{array}
$$

Notice how we put each row a half step to the left of the row above it, so the "centers" are the same.

Filling in the values of these $C(n,k)$ gives:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

This array of $C(n,k)$'s is called Pascal's triangle.

Many of the results about binomial coefficients we've seen before are visible in Pascal's triangle:

- $\sum_{k=0}^{n} C(n,k) = 2^n$ means sum of n^{th} row of Pascal's triangle is 2^n
- $C(n,k) = C(n,n-k)$ means Pascal's triangle is symmetric across vertical axis.

Pattern of even vs. odd entries also very interesting: see the HW problem about this...

The following recurrence for $C(n,k)$ is very useful:

Theorem (Pascal's Identity)

$$C(n+1,k) = C(n,k) + C(n,k-1) \text{ for all } 1 \le k \le n.$$

Note: This means each entry in Pascal's triangle is the sum of the two entries above it:

e.g.
$$\begin{array}{ccccccccccccc} & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \end{array}$$

Together with $C(n,0) = C(n,n) = 1$ on outside this lets us repeatedly fill in all of the triangle.

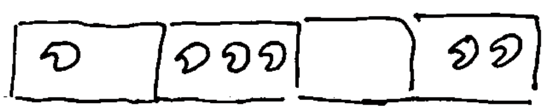
Pf of Pascal's Identity: Let's do a combinatorial proof.

$C(n+1,k)$ is the # of size k subsets of $\{1,2,\ldots,n+1\}$.
Let's show that $C(n,k)+C(n,k-1)$ is also this #.
Let S be size k subset of $\{1,2,\ldots,n+1\}$.
If $n+1 \notin S$, then S is also a size k subset of $\{1,2,\ldots,n\}$, which are counted by $C(n,k)$.
If $n+1 \in S$, then $S \setminus \{n+1\}$ is a size $(k-1)$ subset of $\{1,2,\ldots,n\}$, counted by $C(n,k-1)$.
So there is a bijective correspondence between size k subsets of $\{1,2,\ldots,n+1\}$ (counted by $C(n+1,k)$) and size k or $(k-1)$ subsets of $\{1,2,\ldots,n\}$ (counted by $C(n,k)+C(n,k-1)$ by addition principle).
Thus $C(n+1,k) = C(n,k) + C(n,k-1)$, as claimed. ∎

4/24 The Pigeonhole Principle §6.8

So far we've considered the problem of counting the number of discrete objects satisfying certain conditions. But sometimes we just want to show at least one exists. The Pigeonhole Principle is good for this:

Theorem If you put n pigeons into K holes, and $K < n$, then at least one hole has at least 2 pigeons.

E.g.:  ← 6 pigeons in 4 holes
⇒ at least one hole has at least two pigeons

The trick when using the pigeonhole principle is to figure out what should be the "pigeons" and what the "holes."

E.g.: If there are at least 367 people in a room, then there must be at least two people who share a birthday ("twins").

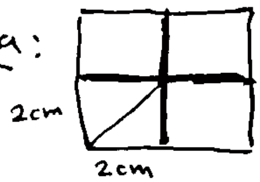
Here the "holes" are the calendar dates, and the "pigeons" are the people.

There are only 366 different dates (remember: Feb. 29) so with 367 people there must be a "collision" of birthdays.

Notice: The Pigeonhole Principle is "non-constructive": it doesn't tell us which people share a birthday or which birthdate is shared...
Also, doesn't necessarily reflect typical behavior:
e.g. with only 23 people, >50% chance of shared birthday, and with 50 people, >97% chance!

E.g. Show that if you put 5 dots on a 4 cm × 4 cm square, at least two dots are within 3 cm of each other.

Idea:



← Break 4 cm × 4 cm square into four 2 cm × 2 cm sub-squares.

Then, by Pigeonhole Principle, at least two of the 5 dots are in same sub-square.

And the maximum distance of two points in a 2 cm × 2 cm square is the length of the diagonal $= 2 \cdot \sqrt{2}$ cm $\approx 2 \cdot 1.4$ cm < 3 cm. ✓

Let's show a more sophisticated example related to divisibility of integers:

Two integers are coprime if they have no common factor (# that divides them) bigger than 1.

E.g. 2 and 6 are not coprime since both divisible by 2.
9 and 15 are not coprime since both divisible by 3.
But 4 and 15 are coprime since they have no common factor.

Theorem If S is a subset of $\{1, 2, 3, \ldots, 20\}$ of size ≥ 11, then there are two numbers a and b in S such that a and b are coprime.

Note: Not true for S of size $= 10$ since
$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$
has all #'s with two as a factor (even #'s), so no two #'s in S are coprime.

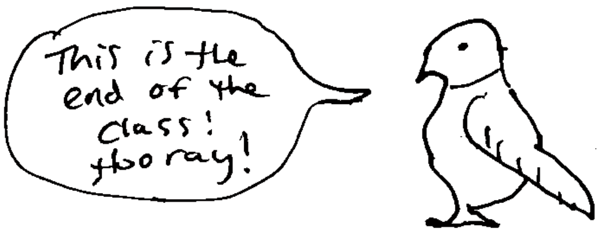
Proof: We first need the following lemma:

Lemma For any positive integer n, the numbers n and $n+1$ are coprime.

Proof: Suppose $r > 1$ is a factor (divisor) of n. Then $n+1 \equiv 1 \mod r$, meaning the remainder when dividing $n+1$ by r is 1. So $n+1$ is not divisible by r. Thus n and $n+1$ have no common factors. $\square$

Next, we use the pigeonhole principle:
Let the "holes" be pairs of consecutive #'s:
$\{1,2\}, \{3,4\}, \{5,6\}, \ldots, \{19, 20\}$
These are 10 holes. So if the subset S has size at least 11, it has two #'s in the same hole. By the previous lemma, those #'s are coprime. $\square$

As you can see from these examples, even though the statement of the pigeonhole principle is very simple, figuring out how to apply it to a given problem can require a lot of creativity, and it can lead to unexpected results!

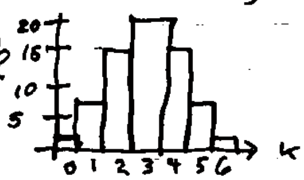


4/26 Further Topics in Discrete Math

We finished the material from the textbook. If you liked this class, here are further topics you could learn about:

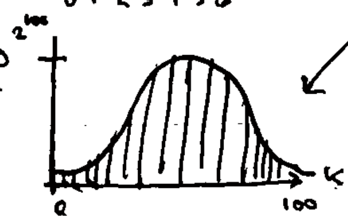
Discrete probability Theory: We talked a little about probabilities with poker hands. One of the most important results in probability is visible in Pascal's triangle: Consider plotting n^{th} row as a histogram:

E.g.: $n=6$



$C(n,k)$ for $k=0$ to n:
1, 6, 15, 20, 15, 6, 1

$n=100$



for big values of n, the histogram will approach a smooth shape called the "Bell curve" or "Gaussian Curve"

This shape tells you how many heads you can expect to see if you flip a fair coin n times. It is a "universal shape" in probability, statistics, and the sciences.

Generating functions: For a sequence of combinatorial numbers, its generating function is a way of recording the sequence in a polynomial or power series:

We have already seen a very important example of a generating function with the Binomial Theorem:

$$(1+x)^n = \sum_{k=0}^{n} C(n,k)\, x^k$$

We keep track of the #'s $C(n,k)$ in the polynomial $(1+x)^n$.

For an ∞ sequence of #'s, we get a power series instead.
Recall the Fibonacci numbers $1, 1, 2, 3, 5, 8, \ldots$
defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

Then: $$\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1 - x - x^2}$$ ← think "Taylor series" from calculus.

This perspective is very powerful in that we can apply techniques from algebra and calculus to understand combinatorial problems:
e.g. radius of convergence is related to growth rate of coefficients.

Graph theory: Graphs consist of vertices (dots •) and edges (u — v) between the vertices.
They are pictures like this:

We have already used graphs to represent functions and relations, but graphs are very versatile structures that can model all kinds of things: e.g. social networks.
There is a lot that can be said about both the typical and extremal structure of graphs!

Thank you all for being excellent students this semester! If you ever want to talk more about math, don't hesitate to send me an email or knock on my office door...