Final Exam Study Guide
Math 181 (Discrete Structures), Spring 2024

1. Sets [§1.1]
   (a) sets of numbers (integers $\mathbb{Z}$ and real numbers $\mathbb{R}$), set-builder notation, subsets ($A \subseteq B$)
   (b) operations of union ($A \cup B$), intersection ($A \cap B$), difference ($A \setminus B$), complement (A^c)
   (c) representing sets via Venn diagrams
   (d) ordered pairs (x, y) and the (Cartesian) product $X \times Y$ of two sets X and Y
2. Logical propositions [§1.2, 1.3]
   (a) operations of "or" ($p \vee q$), "and" ($p \wedge q$), "not" ($\neg p$)
   (b) truth tables for compound propositions
   (c) conditional a.k.a. implication a.k.a. "if... then..." ($p \to q$)
   (d) biconditionals ($p \leftrightarrow q$) and logical equivalence ($\equiv$)
   (e) converse $q \to p$ and contrapositive $\neg q \to \neg p$ of an implication $p \to q$
   (contrapositive is logically equivalent to original implication; converse is not!)
3. Logical arguments [§1.4]
   (a) converting an argument from words to symbolic form and vice-versa
   (b) proving validity using truth tables
   (c) proving validity using the rules of inference and logical equivalences
   (d) common forms of invalid arguments a.k.a. fallacies
4. Quantifiers [§1.5, 1.6]
   (a) propositional formulas ($P(x)$) and domains of discourse (D)
   (b) universal ($\forall x\ P(x)$) and existential ($\exists x\ P(x)$) quantifiers
   (c) DeMorgan's Laws: $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$ and $\neg(\exists x\ P(x)) \equiv \forall x\ \neg P(x)$
   (d) nested quantifiers and order of quantifiers ($\forall x \exists y\ P(x, y) \not\equiv \exists y \forall x\ P(x, y)$)
5. Proofs [§2.1]
   (a) two basic mathematical systems: the theory of integers; the theory of sets
   (b) direct proofs for theorems of form "$\forall x_1, \dots, x_n$ if $P(x_1, \dots, x_n)$ then $Q(x_1, \dots, x_n)$"
   (c) counterexamples to universally quantified statements
6. Indirect proofs [§2.2]
   (a) proof by contrapositive: to prove $p \to q$, prove $\neg q \to \neg p$ instead
   (b) proof by contradiction: assume negation of statement, and deduce contradiction ($r \wedge \neg r$)

7. Mathematical induction [§2.4, 2.5]
   (a) basic structure of inductive proofs: base case $P(1)$, and induction step $P(n) \to P(n+1)$
   (b) proving $\forall(n \in \mathbb{Z}_{>0})\ P(n)$ by induction, especially when $P(n)$ is an algebraic formula
   (c) finding patterns to guess formulas involving n which can then be proved by induction
8. Functions [§3.1]
   (a) ways to view a function $f\colon X \to Y$: rule to convert input $x \in X$ to output $y = f(x) \in Y$; set of ordered pairs (x, y); arrow diagram from X to Y
   (b) one-to-one, onto, and bijective functions
   (c) composition of functions, and inverse functions
   (d) modular arithmetic functions like $f(x) = x \mod n$
9. Sequences and strings [§3.2]
   (a) finite and infinite sequences: ordered list of elements of some set
   (b) set of strings X^* on a finite alphabet X, the null string $\lambda \in X^*$, concatenation of strings
   (c) subsequences (not necessarily consecutive) versus substrings (consecutive)
10. Relations [§3.4, 3.5]
    (a) digraph representation of a relation R on a set X
    (b) properties that R can have: reflexive, symmetric, anti-symmetric, transitive
    (c) partial order (reflexive, anti-symmetric, transitive): way to "compare" things in X
    (d) equivalence relation (reflexive, symmetric, transitive): way to say certain things in X are "the same"; corresponds to a partition of X into equivalence classes
11. Basic counting principles [§6.1]
    (a) multiplication principle: total # of possibilities = product of # of choices at each step
    (b) addition principle: size of union of *disjoint* sets is sum of sizes of the sets
    (c) principle of inclusion and exclusion: $\#(X \cup Y) = \#X + \#Y - \#(X \cap Y)$
12. Permutations and combinations [§6.2, 6.3]
    (a) number of permutations (=orderings) of n element set is $n! = n \times (n-1) \times \cdots \times 1$, and number of k-permutations (=orderings of k element subsets) is $P(n, k) = n!/(n-k)!$
    (b) for rearrangements of word with repeated letters like MISSISSIPPI use $n!/(k_1!k_2! \cdots k_m!)$
    (c) number of k-combinations (= k element subsets) of n element set is the binomial coefficient, a.k.a. "n choose k" number, $C(n, k) = n!/(k! \cdot (n-k)!)$
    (d) for selections of k things from n things with repeats allowed use $C(k+n-1, k)$
13. Binomial coefficients [§6.7]
    (a) Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} C(n, k) x^k y^{n-k}$
    (b) Pascal's Triangle of $C(n, k)$, defined by recurrence $C(n+1, k) = C(n, k) + C(n, k-1)$
14. Pigeonhole principle [§6.8]
    (a) if you place n pigeons in k holes, with $k < n$, at least one hole has at least two pigeons