Math 210 (Modern Algebra I), Final Exam: Part 1,

Fall 2024; Instructor: Sam Hopkins; Due on: Wednesday, December 4th

This is the take-home part of the final. You have one week to work on these questions, and may consult your notes and the textbook, but not other students. Each problem is worth 10 points.

1. Recall that the *dihedral group* D_4 has presentation $D_4 = \langle r, s \mid r^4 = s^2 = (rs)^2 = e \rangle$. And the *quaternion group* Q_8 has presentation $Q_8 = \langle \overline{e}, i, j, k \mid \overline{e}^2 = e, i^2 = j^2 = k^2 = ijk = \overline{e} \rangle$.
   (a) Show that both D_4 and Q_8 are groups of order 8 by listing the 8 elements in each group.
   (b) Show that D_4 and Q_8 are both *not* abelian by finding, in each group, a pair of elements a and b with $ab \neq ba$.
   (c) Prove that D_4 and Q_8 are *not* isomorphic.
2. Let the group G act on a set S. Recall that we say the action is *transitive* if for every $x, y \in S$ there exists a $g \in G$ such that $gx = y$. And we say the action is *faithful* if the only element that acts as the identity is the identity element, i.e., $gx = x$ for all $x \in S$ implies that $g = e$. Suppose that the action of G on S is transitive and faithful.
   (a) Let $x \in S$ and let G_x be its stabilizer. Show that if N is a normal subgroup of G with $N \subseteq G_x$ then $N = \{e\}$.
   (b) Now suppose further that G is abelian and finite. Conclude that $|S| = |G|$.
3. Let R be a ring (not necessarily commutative, but with 1) such that for every nonzero $a \in R$, there exists a *unique* $b \in R$ with $aba = a$.
   (a) Prove that R has no nonzero zero divisors.
   (b) Prove that every nonzero $a \in R$ is a unit (i.e., has a multiplicative inverse).
4. Let R be a commutative ring, A an R-module, and $f\colon A \to A$ an R-module homomorphism. Suppose that $f \circ f = f$ (where $\circ$ denotes composition). Prove that $A = \ker(f) \oplus \operatorname{im}(f)$. **Hint**: this is the same as saying that each $a \in A$ can be written uniquely as $a = b + c$ with $b \in \ker(f)$ and $c \in \operatorname{im}(f)$.