Math 210 (Modern Algebra I), HW# 2,

Fall 2024; Instructor: Sam Hopkins; Due: Wednesday, September 11th

1. Prove that the nontrivial groups with no proper, nontrivial subgroups are $\mathbb{Z}/p\mathbb{Z}$ for p prime.

2. For a positive integer n, the *multiplicative group* $(\mathbb{Z}/n\mathbb{Z})^\times$ consists of those $a \in \mathbb{Z}/n\mathbb{Z}$ satisfying $\gcd(a, n) = 1$ (i.e., coprime to n), with product given by multiplication modulo n. This group is *not* the same as the additive group $\mathbb{Z}/n\mathbb{Z}$: e.g., the identity element in $(\mathbb{Z}/n\mathbb{Z})^\times$ is 1.

   Now let p be a prime. Use Lagrange's Theorem for the group $(\mathbb{Z}/p\mathbb{Z})^\times$ to prove *Fermat's Little Theorem*, which states that $a^p \equiv a \mod p$ for all $a \in \mathbb{Z}$.

3. (a) Let G be a (not necessarily finite!) group and $H \leq G$ a subgroup of G with $[G : H] = 2$. Prove that H is a normal subgroup of G.

   (b) Give an example of a group G and a subgroup $H \leq G$ with $[G : H] = 3$ such that H is not a normal subgroup of G.

4. Let D_n denote the dihedral group of symmetries of a regular n-gon. Prove that the map $\varphi\colon D_n \to \mathbb{Z}/2\mathbb{Z}$ which sends all reflections to 1 and all other elements to 0 is a homomorphism. Explain why the kernel of φ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

5. Again letting D_n denote the dihedral group, recall that in class we showed that D_n has a presentation $D_n = \langle r, s\colon r^n = s^2 = (sr)^2 = 1\rangle$, where r corresponds to clockwise rotation by $\frac{2\pi}{n}$ radians and s corresponds to one of the reflections. Explain why we also have the presentation $D_n = \langle s, t\colon s^2 = t^2 = (st)^n = 1\rangle$.