Math 210 (Modern Algebra I), HW# 3,

Fall 2024; Instructor: Sam Hopkins; Due: Wednesday, September 25th

1. For p a prime number, a group G is called a *p-group* if every element has order a power of p. Prove that a finite abelian p-group is generated by its elements of maximal order.

2. Let G be a group. An automorphism $\varphi \in \mathrm{Aut}(G)$ is called *inner* if it is conjugation by some fixed $h \in G$, i.e., is of the form $\varphi\colon g \mapsto hgh^{-1}$. Also recall that the *center* of G is $Z(G) = \{g \in G\colon gx = xg \text{ for all } x \in G\}$.

   (a) Prove that $\mathrm{Inn}(G)$, the set of inner automorphisms of G, is a subgroup of $\mathrm{Aut}(G)$. (In fact it is a normal subgroup, but you do not need to prove that.)

   (b) Prove that $Z(G)$ is a normal subgroup of G.

   (c) Prove that $G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$.

3. An action of a group G on a set S is called *transitive* if for every $x, y \in S$ there is a $g \in G$ such that $g \cdot x = y$. An action of a group G on a set S is called *free* if $g \cdot x = x$ for some $x \in S$ and $g \in G$ implies $g = e$. In what follows, let $S = \{1, 2, \ldots, n\}$ and let G be a finite group.

   (a) Suppose G acts transitively on S. Prove that n divides the order of G.

   (b) Suppose G acts freely and transitively on S. Prove that the order of G is exactly n.

   (c) Give an example, for each $n \geq 1$, of such a G acting freely and transitively on S.

4. The *cycle type* of a permutation $\sigma \in S_n$ in the symmetric group on n letters is the list $m_1(\sigma), m_2(\sigma), \ldots, m_n(\sigma)$ where $m_i(\sigma)$ is the number of i-cycles in σ's cycle decomposition.

   (a) Prove that two permutations in S_n are in the same conjugacy class if and only if they have the same cycle type.

   (b) Prove that the cardinality of the conjugacy class of $\sigma \in S_n$ is $\dfrac{n!}{1^{m_1}\, m_1!\, 2^{m_2}\, m_2! \, \cdots \, n^{m_n}\, m_n!}$ where $m_i = m_i(\sigma)$ are the numbers in the cycle type of σ.

5. Let G be a finite group of order pq for distinct primes $p < q$. Prove that G is not simple, i.e., that it has a normal subgroup $N \trianglelefteq G$ other than $\{e\}$ and G.
   Hint: use the Sylow theorems; specifically, show that any Sylow q-subgroup is normal in G.