Math 210 (Modern Algebra I), HW# 5,

Fall 2024; Instructor: Sam Hopkins; Due: Wednesday, October 30th

1. Let $R = \mathbb{Z}[\sqrt{-5}]$ be the subring of complex numbers of the form $a + b\sqrt{-5}$ with $a, b \in \mathbb{Z}$.
   (a) Define a *norm* $N\colon R \to \{0, 1, \ldots\}$ on this ring by $N(a + b\sqrt{-5}) = a^2 + 5b^2$. Show that this norm is *multiplicative*, i.e., that $N(xy) = N(x)N(y)$ for $x, y \in R$.
   (b) Show that $N(x) = 0 \Leftrightarrow x = 0$ and $N(x) = 1 \Leftrightarrow x$ is a unit, for $x \in R$.
   (c) Using this norm N, show that the elements 2, 3, $1+\sqrt{-5}$, $1-\sqrt{-5} \in R$ are irreducible.
   (d) Show 2, 3, $1+\sqrt{-5}$, $1-\sqrt{-5} \in R$ are not prime. **Hint**: $2 \cdot 3 = 6 = (1+\sqrt{-5})(1-\sqrt{-5})$.

   (Note that (c) and (d) imply that R is not a unique factorization domain.)

2. Consider the polynomial ring $R = \mathbb{Z}[x]$ and the ideal $I = \langle 2, x \rangle \subseteq R$. Show that I is not a principal ideal. (Hence R is not a principal ideal domain.)

3. Given an example of a polynomial $f(x)$ whose coefficients all belong to $\{0, 1\}$ such that:
   - $f(x)$ is irreducible when viewed as an element of $\mathbb{Z}[x]$;
   - $f(x)$ is *reducible* when viewed as an element of $\mathbb{F}_2[x]$, where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

4. (a) Prove that $1 + x$ is a unit in the formal power series ring $\mathbb{Z}[[x]]$.
   Hint: think about the Taylor series expansion of $\frac{1}{1+x}$.
   (b) Let R be a (not necessarily commutative!) ring. Recall that an element $x \in R$ is called *nilpotent* if there is some $n \geq 1$ such that $x^n = 0$. Prove that if $x \in R$ is nilpotent, then $1 + x$ is a unit of R. **Hint**: use a similar strategy as in part (a).

5. Recall that a *local ring* is a commutative ring R with a unique maximal ideal $\mathfrak{m} \subseteq R$. The quotient $R/\mathfrak{m}$ is called the *residue field* of the local ring R. Now let p be a prime number, and consider $R = \mathbb{Z}_{(p)} = \{\frac{a}{b} \colon a, b \in \mathbb{Z}, p \nmid b\}$, the integers localized at the prime ideal (p). What is the residue field of this R? Explain your answer.