Math 210 (Modern Algebra I), HW# 6,

Fall 2024; Instructor: Sam Hopkins; Due: Wednesday, November 13th

1. Let R be a ring (not necessarily commutative, but with 1) and let M be left R-module.
   (a) For $x \in M$, define the *annihilator* of x to be $\mathrm{Ann}(x) := \{r \in R \colon rx = 0\}$. Prove that $\mathrm{Ann}(x)$ is always a left ideal of R.
   (b) Suppose that M is cyclic, i.e., $M = \langle x \rangle$ for some $x \in M$. Prove that $M \simeq R/\mathrm{Ann}(x)$.
2. Let R be a commutative ring.
   (a) Prove that the polynomial ring $R[x]$ is naturally an R-module.
   (b) Prove that the following is a short exact sequence of R-modules:
   $$0 \to R[x] \xrightarrow{\cdot x} R[x] \to R \to 0$$
   Here $R[x] \xrightarrow{\cdot x} R[x]$ is the map $f(x) \mapsto x \cdot f(x)$, and $R[x] \to R$ is the map $f(x) \mapsto f(0)$.
3. Let p be a prime number and A an abelian group. Show that $A[p] := \{a \in A \colon pa = 0\}$ is naturally a vector space over $\mathbb{Z}/p\mathbb{Z}$. Deduce that if A is finite, then $|A[p]| = p^n$ for some n.
4. Let $m \geq 1$ be a positive integer.
   (a) Prove that for any abelian group A, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, A) \simeq A[m] := \{a \in A \colon ma = 0\}$.
   (b) Use part (a) to show that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \simeq \mathbb{Z}/\gcd(m,n)\mathbb{Z}$ for any $n \geq 1$.
   (c) Use part (a) to show that the dual $(\mathbb{Z}/m\mathbb{Z})^*$ of $\mathbb{Z}/m\mathbb{Z}$, as a $\mathbb{Z}$-module, is 0.
5. (a) Explain why the sequence $0 \to \mathbb{Z} \to \mathbb{Q}$ of abelian groups is exact.
   (b) Prove that, after tensoring over $\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$, the induced sequence of abelian groups
   $$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$$
   is *not* exact.

   (This means that "the tensor product functor is not left-exact.")