Math 210 (Modern Algebra I), Midterm # 1,

Fall 2024; Instructor: Sam Hopkins; Taken on: Wednesday, October 2nd

Each problem is worth 10 points, for a total of 50 points. You have 80 minutes to do the exam. Partial credit will be given generously, so write as much as you know for each problem.

1. Give an example of two finite groups G and H of the same order which are not isomorphic. Explain why your example is correct.

2. Let $n \geq 1$ be a positive integer and recall the dihedral group $D_n = \langle r, s \colon r^n = s^2 = (rs)^2 = 1 \rangle$ is the group of symmetries of a regular n-gon, where r is clockwise rotation by $\frac{2\pi}{n}$ radians and s is a reflection. Now suppose that $n = 2m$ is even and let $N = \langle r^m \rangle \leq D_n$.

   (a) Prove that N is a normal subgroup of D_n.

   (b) What is the order of the quotient group D_n/N?

3. (For this problem, recall the notations $n\mathbb{Z} = \{nx \colon x \in \mathbb{Z}\}$, $G \cap H = \{x \colon x \in G \text{ and } x \in H\}$ and $G + H = \{g + h \colon g \in G, h \in H\}$.) Consider the subgroups $G = 15\,\mathbb{Z}$ and $H = 20\,\mathbb{Z}$ of $\mathbb{Z}$, the integers under addition. Define the numbers m_1, m_2, m_3, m_4 by

   $$G/(G\cap H) \simeq \mathbb{Z}/m_1\mathbb{Z}; \quad H/(G\cap H) \simeq \mathbb{Z}/m_2\mathbb{Z}; \quad (G+H)/G \simeq \mathbb{Z}/m_3\mathbb{Z}; \quad (G+H)/H \simeq \mathbb{Z}/m_4\mathbb{Z}.$$

   What are m_1, m_2, m_3, and m_4? **Hint**: the 2nd isomorphism theorem can save you time here.

4. Fix positive integers $1 \leq k \leq n$. Let $\mathcal{F}$ denote the set of k-element subsets of $\{1, 2, \ldots, n\}$ and let $G = S_n$, the symmetric group on n letters, act on $\mathcal{F}$ by setting $\sigma \cdot X = \{\sigma(i) \colon i \in X\}$ for all $X \in \mathcal{F}$ and $\sigma \in G$. Now fix any one $X \in \mathcal{F}$, e.g., $X = \{1, \ldots, k\}$.

   (a) Describe the orbit of X under G.

   (b) Describe the stabilizer $G_X \leq G$.

   (c) Use the orbit-stabilizer theorem to prove that $|\mathcal{F}| = \frac{n!}{k!(n-k)!}$.

5. Let p be a prime number and let $G = S_p$ be the symmetric group on p letters.

   (a) Explain why the Sylow p-subgroups of G are $\langle \sigma \rangle$ for $\sigma \in G$ a p-cycle.

   (b) Explain why this means that n_p, the number of Sylow p-subgroups of G, is $\frac{1}{p-1}$ times the total number of p-cycles in G.

   (c) Explain why the total number of p-cycles in G is $(p-1)!$.

   (d) Use the Sylow theorems to conclude that $(p-2)! \equiv 1 \mod p$. (This is *Wilson's theorem*.)