Math 210 (Modern Algebra I), Midterm # 2,

Fall 2024; Instructor: Sam Hopkins; Taken on: Wednesday, November 20th

Each problem is worth 10 points, for a total of 50 points. You have 80 minutes to do the exam. Partial credit will be given generously, so write as much as you know for each problem.

1. Give a specific example of a noncommutative ring R (with a 1). Show that your example really is noncommutative by exhibiting two elements $x, y \in R$ with $xy \neq yx$.

2. Recall that the ring of *Gaussian integers* is $\mathbb{Z}[i] = \{a + bi \colon a, b \in \mathbb{Z}\}$ where $i = \sqrt{-1} \in \mathbb{C}$. Consider the ideal $I = \langle 2, 3 + i \rangle \subseteq \mathbb{Z}[i]$. We proved that $\mathbb{Z}[i]$ is a principal ideal domain. Hence there exists an $r \in \mathbb{Z}[i]$ for which $I = \langle r \rangle$. Find such an r.
   Hint: how can the identity $(1 - i)(1 + i) = 2$ help you?

3. (For this problem, recall $\mathbb{R}[x]$ is the one variable polynomial ring over the real numbers $\mathbb{R}$.)
   (a) State what it means for an element $p \in R$ of a commutative ring R to be prime.
   (b) Is $x^2 - 1$ a prime element of $\mathbb{R}[x]$? Justify your answer.
   (c) Is $x^2 + 1$ a prime element of $\mathbb{R}[x]$? Justify your answer.

4. Consider the following sequence of abelian groups and homomorphisms between them:

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \to 0.$$

   Here $f \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by $f(a) = (a, a)$, and $g \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ is given by $g(a, b) = a - b$. Prove that this is a short exact sequence.

5. Recall for a commutative ring R and an R-module M, its *dual module* is $M^* = \mathrm{Hom}_R(M, R)$. Give a specific example of a commutative ring R and an R-module M such that M is not isomorphic to its double dual M^{**}. Explain why your example works.