2/5

Splitting fields and normality § 5.3

The Fund. Thm. of Galois Theory is a very powerful result, but it requires the assumption that the extension L/K is Galois, and as we defined Galois, to check it requires a precise understanding of how $Aut_K(L)$ acts on L. It would be preferable to have a more "instringic" field criterion...

Def'n Let K be a field, L an extension of K, and $f(x) \in K[x]$ a poly. We say that $f(x)$ splits in L if $f(x) = u_0(x-u_1)(x-u_2)\cdots(x-u_n)$ with $u_i \in L$, i.e., $f(x)$ factors completely into linear factors over L. We say that L is a splitting field of $f(x)$ if $f(x)$ splits over L and $L = K(u_1, \ldots, u_n)$ where these u_i are all the roots of $f(x)$.

E.g. With base field $K = \mathbb{Q}$, $L = \mathbb{Q}(\sqrt{2})$ is a splitting field of $f(x) = x^2 - 2$ since all its roots (namely $\sqrt{2}$ and $-\sqrt{2}$) live in L. But $L = \mathbb{Q}(\sqrt[3]{2})$ is not a splitting field of $f(x) = x^3 - 2$, since not all its roots lie in L (we're missing $\omega\sqrt[3]{2}$ and $\omega^2\sqrt[3]{2}$).

As we will see, splitting fields of irreducible polynomials are basically how we produce Galois extensions. We need a few more defs...

Def'n Let $f(x) \in K[x]$ be an irreducible polynomial. We say it is separable if in any (or every) splitting field L of $f(x)$, $f(x)$ splits into distinct factors $f(x) = u_0(x-u_1)\cdots(u_n)$, i.e., $u_i \neq u_j$ for $i \neq j$.

Remark Suppose $f(x)$ is a min. poly. of $u \in L$ (hence irreducible) but not separable because it has a double root of u:
$f(x) = (x-u)^2 q(x)$. We can take its derivative

$Df(x)$ (defined formally), and from the product rule we will see that $(x-u)$ also divides $Df(x)$, i.e., u is a root of $Df(x)$. But $Df(x)$ has lower degree than $f(x)$, which was supposed to be the minimal polynomial of u! The only way this can happen without a contradiction is if $Df(x) = 0$!
Over a field of characteristic zero (like $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, etc.) the derivative cannot be zero, so in char. 0 we never have to worry about separability! We'll come back to positive characteristic later...

Def'n An algebraic extension L of K is called separable if for every $u \in L$ the min. poly. of u in $K[x]$ is a separable polynomial.

Def'n An algebraic extension L of K is called normal if for every irreducible polynomial $f(x) \in K[x]$, whenever f has at least one root in L, then in fact $f(x)$ splits completely in L.

Thm An algebraic extension L/K is Galois if and only if it is both separable and normal.

Pf sketch: Let's prove alg. & Galois $\Rightarrow$ separable & normal. See the book for the other direction. Let $u \in L$, and let $u_1, \ldots, u_n \in L$ be the distinct roots of the min. poly $f(x) \in K[x]$ which lie in L. Form the polynomial $g(x) = (x-u_1)(x-u_2)\cdots(x-u_n) \in L[x]$. For any $\sigma \in \mathrm{Aut}_K(L)$, σ permutes the u_i in some way, so $\sigma(g(x)) = g(x)$, i.e., σ acts trivially on the coefficients of $g(x)$. But since L/K is Galois, this must mean all the coefficients of $g(x)$ are in K, i.e. $g(x) = f(x)$ is the min. poly. of u, which thus splits into distinct factors in L. So indeed L/K is normal & separable! $\boxtimes$

In the case of a <u>finite</u> extension, we can do even better.

<u>Thm</u> (Artin) Let L/K be a finite extension. The following are equivalent:

i) L/K is Galois,

ii) L/K is the splitting field of a polynomial $f(x) \in K[x]$ all of whose irreducible factors are separable,

iii) $[L:K] = |\mathrm{Aut}_K(L)|$.

<u>Pf</u>: Similar to what we have seen, see book for details $\boxtimes$

So indeed in char. 0, finite Galois extensions are exactly splitting fields of polynomials.

<u>Algebraic closures</u> § 5.3

Def'n A field L is called <u>algebraically closed</u> if every poly. $f(x) \in L[x]$ splits in L. An algebraic extension L of a field K is called ~~the~~ <u>algebraic closure</u> of K if every poly. $f(x) \in K[x]$ splits in L, equivalently, is L is alg. closed

<u>Thm</u> Every field K has an algebraic closure, unique up to isomorphism.

<u>Pf</u>: This is quite nontrivial but I am skipping it - see book! $\boxtimes$

<u>E.g.</u> The algebraic closure of $\mathbb{R}$ is $\mathbb{C} = \mathbb{R}(i)$, which is basically equivalent to the "Fund. Thm. of Algebra."

<u>E.g.</u> The algebraic closure of $\mathbb{Q}$, denoted $\mathbb{Q}^{alg}$ or $\overline{\mathbb{Q}}$, is the set of all "<u>algebraic numbers</u>". Things like $\sqrt{2}$, $\sqrt[3]{5+\sqrt{2}}$, $i=\sqrt{-1}$ and so on live in $\mathbb{Q}^{alg}$. But "most" real numbers, including π and e (transcendental!) do not belong to $\mathbb{Q}^{alg}$. In fact, $\mathbb{Q}^{alg}$ is "countably infinite", unlike $\mathbb{R}$ or $\mathbb{C}$.

2/10

Finite Fields §5.5

Def'n Let K be a field. The characteristic of K is the smallest $n \geq 1$ such that $n \; (= \overbrace{1+1+\cdots+1}^{n \text{ times}}) = 0$ in K, or is zero if no such n exists.

E.g. most of the fields we have seen so far, like $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ (and their extensions) have characteristic zero. For an example of a field with "positive characteristic", recall that for a prime number p we have the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, which has characteristic p.

Prop. The characteristic of a field K is 0 or a prime number p.

Pf sketch: Suppose the characteristic of K were $n > 0$ a non-prime number, e.g. $n = 6$. Take any proper divisor of n, e.g. $d = 2$. Then $2 = 1+1$ is a non-zero zero divisor in K, so K cannot be an integral domain (much less a field). ☒

Def'n Let K be a field. The intersection of all subfields of K is called the prime subfield of K. It is the "smallest" subfield in K.

Prop. The prime subfield of K is either $\mathbb{Q}$, if K has char. 0, or $\mathbb{F}_p$, if K has positive char. $p > 0$.

Pf: The prime subfield of K is the one generated by $1 \in K$. If K has char. p so that $p \cdot 1 = \overbrace{1+1+\cdots+1}^{p}$ then this will be $\mathbb{F}_p$, otherwise we will get a copy of $\mathbb{Z}$, hence $\mathbb{Q}$, inside K. ☒

Corollary If K is a finite field, then it must have positive characteristic.

Pf: otherwise it would have $\mathbb{Q}$ inside it, which is infinite. ☒

Remark Every finite field has positive characteristic, but the converse is not true: there are infinite fields of char. $p>0$. For example, $K=\mathbb{F}_p(x)$, field of rational functions with coefficients in $\mathbb{F}_p$, is infinite of characteristic p. So is $K=\overline{\mathbb{F}_p}$, algebraic closure of $\mathbb{F}_p$ (we may discuss this later).

In fact, we can say a little more about how finite fields look:

Prop. Let K be a finite field. Then the number of elements in K is p^n, where p is the char. of K, for some $n \geq 1$.

Pf: The prime subfield of K is $\mathbb{F}_p$ and K is a finite dimensional v.s. over this $\mathbb{F}_p$, hence has p^n elts where n is its dimension as an $\mathbb{F}_p$-vector space. ☒

In what follows we will show that, for any prime power $q=p^n$, a finite field $\mathbb{F}_q$ exists and is unique! But be warned that while $\mathbb{F}_p=\mathbb{Z}/p\mathbb{Z}$ is very easy to construct, constructing $\mathbb{F}_q$ for q a prime power which is not a prime is much more involved! In particular...

Note For $n>1$, $\mathbb{F}_{p^n}$ is not the same as $\mathbb{Z}/p^n\mathbb{Z}$. Indeed, for any composite number N, $\mathbb{Z}/N\mathbb{Z}$ is not an integral domain, hence not a field!

To construct finite fields $\mathbb{F}_q$ for $q=p^n$ with $n>1$, we will instead realize them as (algebraic!) extensions of $\mathbb{F}_p$. Hence, our study of field extensions and Galois groups etc. is very useful for this purpose. Sometimes finite fields are called "Galois fields" for this reason...

One of the best tools for studying fields of positive characteristic is the Frobenius endomorphism (or automorphism).

Thm Let K be a field of char. $p>0$. Define the map $\varphi: K \to K$ by $\varphi(x) = x^p$ for all $x \in K$. Then φ is a $\mathbb{F}_p$-linear endomorphism of K (i.e., it preserves $\mathbb{F}_p$ and the field structure of K). It is called the Frobenius endomorphism. It is always injective. If K is finite, it is also surjective, called the Frobenius automorphism.

Pf: We need to check that φ preserves the field operations. That it preserves multiplication (& division) is clear: $\varphi(xy) = (xy)^p = x^p y^p$. The important thing to check is that it preserves addition. Recall the Binomial Theorem $(x+y)^p = \sum_{i=0}^{p} \binom{p}{i} x^i y^{p-i}$, where $\binom{p}{i} = \frac{p!}{i!(p-i)!}$ are the binomial coefficients. Notice that for $0<i<p$, $\frac{p!}{i!(p-i)!}$ (an integer) has a factor of p on top that never cancels, hence modulo p we have $\binom{p}{i} = 0$ for these i, which means that $(x+y)^p = x^p + y^p$ (sometimes called the "Freshman's Dream.") So indeed φ preserves addition. It acts as the identity on $\mathbb{F}_p$, the prime subfield of K, since $\varphi(1) = 1$. It is injective since $\varphi(x) \neq 0$ for any $x \neq 0$ since K has no non-zero zero divisors. If K is finite, it's bijective since an injective map between two finite sets of the same size is bijective. $\boxtimes$

Remark: The Frobenius endomorphism is not always a bijection. For example, with $K = \mathbb{F}_p(x)$ it fails to be surjective. A field K is called perfect if it either has characteristic zero, or has positive char. $p>0$ and the Frobenius endomorphism is surjective. This is the same as every irreducible polynomial $f(x) \in K[x]$ being separable. (See also the last problem on your HW...).

2/12

<u>Def'n</u> If K is a finite field, [redacted] its <u>order</u> is its size, i.e., $\# K$.

We will see that if K is a finite field of char. p, then the Frobenius automorphism φ <u>generates</u> the Galois group $\mathrm{Aut}_{\mathbb{F}_p}(K)$. First, let's start with the <u>multiplicative group</u>:

<u>Thm</u> Let K be a finite field of order $q = p^n$. Then its <u>multiplicative group</u> $(K \setminus \{0\}, \times)$ is <u>cyclic</u> (of order $q-1$).

<u>Pf</u>: The multiplicative group, whatever it is, is some finite abelian gp., hence by classification has form $\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_m\mathbb{Z}$ where $d_1 | d_2 | \cdots | d_m$. We see that for any $g \in G$ (where G is this gp.) we have $d_m \cdot g = 0$ in additive notation. Multiplicatively, we can say $x^{d_m} - 1 = 0$ for all $x \in K \setminus \{0\}$. But $\# K \setminus \{0\} = q-1$, which is the biggest that d_m could be (if G were cyclic), and a polynomial can have at most as many <u>roots</u> as its degree, so in fact $d_m = q-1$, $m = 1$, and G is cyclic! $\blacksquare$

<u>Remark</u>: In general, finding a <u>generator</u> of the mult. group of a finite field can be a difficult computational problem. The number of generators is $\Phi(q-1)$ where Φ is "<u>Euler's totient function</u>" $\phi(n) = \# \{k \leq n : \gcd(n,k) = 1\}$.

<u>Thm</u> For any prime power $q = p^n$, a finite field of order q exists, and all such finite fields are isomorphic: it is the <u>splitting field</u> of $f(x) = x^{p^n} - x$ over $\mathbb{F}_p$.

<u>Pf</u>: First we address uniqueness, so let K be a finite field of order p^n. As we just explained $x^{p^n - 1} - 1 = 0$ for all $x \in K$, $x \neq 0$. Hence, $x^{p^n} - x = 0$ for all $x \in K$. So indeed the poly. $f(x) = x^{p^n} - x = \prod_{u \in K} (x - u)$ splits in K. And since the roots of this polynomial are all of K, K is the splitting field

Now we deal with existence. By looking at the formal derivative of $f(x) = x^{p^n} - x$ (which is -1 mod p) we can see that in a splitting field of $f(x)$ it has all distinct roots, i.e. is separable. So let K be a splitting field of $f(x)$ and let $E \subseteq K$ be the set of roots of $f(x)$ in K. Then $\# E = p^n$. But also, $E = \{u \in K : \varphi^n(u) = u\}$ where $\varphi : K \to K$ is the Frob. auto., hence E is a subfield (fixed points of an automorphism), and since E contains all roots of $f(x)$, we must have $K = E$. $\boxtimes$

Remark: Something we have yet to formally address, implicit in the above proof, is that for any field K and any poly. $f(x) \in K[x]$, a splitting field of $f(x)$ exists and it is unique. This can be established in the following way. First:

Lemma 1) If $f(x) \in K[x]$ is irreducible, then there is a simple algebraic extension $K(u)$ where the min. poly. of u is $f(x)$.

2) If $K(u)$ and $K(v)$ are two simple algebraic extensions s.t. the min. poly.'s of u and v are the same, they are isomorphic.

Pf: For 1): Take $K[X] / \langle f(x) \rangle$ as our field.

For 2): $\psi : K(u) \to K(v)$ defined by $\psi(u) = v$ is the iso. $\boxtimes$

Then, to construct a splitting field of $f(x)$ over K, we inductively factor $f(x)$ into irreducibles and adjoin roots of the irreducible factors of degree 2 or higher until it completely factors. Part 2) of the above lemma can also be used to show that this process results in a unique field independent of what choice of roots we adjoin and in what order. So indeed the field $\mathbb{F}_q$ with $q = p^n$ elts. exists & is unique.

$\mathbb{F}_{p^n}$ as an extension of $\mathbb{F}_p$ is Galois.

Cor The Galois group $\mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^n})$ is cyclic of order n, generated by the Frobenius automorphism φ.
For each divisor $d \mid n$, there is a unique subfield $\mathbb{F}_{p^d}$ in $\mathbb{F}_{p^n}$.

Pf: By the above discussion, any subfield $\mathbb{F}_{p^k}$ will be the fixed points of the k^{th} power of φ, hence indeed $\mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^n})$ is generated by φ. (To show $\mathbb{F}_{p^n}/\mathbb{F}_p$ is Galois, note it is the splitting field of a sep. polynomial)
The last sentence follows from the Fund. Thm. of Galois Theory.

Cor Let $f(x) \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree n, and let $K = \mathbb{F}_p(u)$ where u has minimal polynomial $f(x)$.
Then $K = \mathbb{F}_{p^n}$. Pf: The degree $[K : \mathbb{F}_p] = n$, so we have $\# K = p^n$ and by uniqueness of finite fields this means $K = \mathbb{F}_{p^n}$. ☒

Remark: In practice, to construct $K = \mathbb{F}_{p^n}$ we find an irreducible polynomial $f(x) \in \mathbb{F}_p[x]$ of deg. n and adjoin a root of it to $\mathbb{F}_p$.
Because to work algorithmically in this K we need to use polynomial long division and the Euclidean gcd algorithm, it is preferable to choose such an $f(x)$ where most coeff's $= 0$.
For example, taking $f(x) = x^4 + x + 1 \in \mathbb{F}_2[x]$ works to construct $\mathbb{F}_{16} = \mathbb{F}_2[x]/\langle x^4 + x + 1 \rangle$ in this way.
But cannot always choose $f(x) = x^n + x + 1$:
e.g. see exercise 9 of section 5.5 of the textbook.
One choice of irreducible polynomials over finite fields are the "Conway polynomials" but they are slightly complicated to describe ...

2/19

The Galois group of a polynomial §5.4

Having finished our brief tour of the world of characteristic p and finite fields, we return to studying finite extensions of $\mathbb{Q}$. Because the Fund. Thm. of Galois Theory is a very powerful tool for studying these extensions, we will focus on finite Galois extensions of $\mathbb{Q}$, which we now know are the same as splitting fields of polynomials. (In particular, since we are most interested in char. 0, we will largely ignore issues of separability...)

Def'n Let K be a field and $f(x) \in K[x]$ a polynomial. The Galois group of $f(x)$ is $\mathrm{Aut}_K(L)$ where L is a splitting field of f.

Remark: Recall that we sketched an argument for why splitting fields of any $f(x)$ exist and are 'unique'.

Note: In what follows we will make the assumptions that:
- all polynomials $f(x)$ under consideration are monic,
- all poly's $f(x)$ have all irreducible factors that are separable.

(The monic assumption is harmless b.c. we can always divide by leading coeff., and separability always holds in char. 0).

With these assumptions, in its splitting field L the polynomial factors as $f(x) = \prod_{i=1}^{n} (x-u_i)$ with $u_1, \ldots, u_n \in L$ the distinct roots of f.

Thm The Galois group of $f(x)$ is a subgroup of the symmetric group S_n acting on the roots u_i by permutation. If $f(x)$ is irreducible, then this subgroup/action is transitive, where we recall that transitive means for every u_i, u_j there is some $\sigma \in G = \mathrm{Aut}_K(L)$ such that $\sigma(u_i) = u_j$.

We usually view Galois groups of polynomials as permutation groups in this way...

Pf sketch of thm: We know that any $\sigma \in \mathrm{Aut}_K(L)$ permutes the roots of any ~~irreducible~~ polynomial like $f(x)$, but since $L = K(u_1, \ldots, u_n)$ is generated by these roots, it is determined by this permutation. To see that if $f(x)$ is irreducible, then this action is transitive, note that then f must be the minimal polynomial of the u_i, so indeed a mapping $\sigma(u_i) = u_j$ can always be extended to an automorphism $\sigma \in \mathrm{Aut}_K(L)$. ⊠

"Generically", the Galois group of an irreducible polynomial $f(x)$ of degree n will be the full symmetric group S_n, but for "special" $f(x)$ it can be smaller.

E.g. Consider $f(x) = x^4 + x^3 + x^2 + x + 1$. Notice that $(x-1) f(x) = x^5 - 1$, so the roots of $f(x)$ are the 5th roots of unity other than 1, i.e., $\omega, \omega^2, \omega^3$, and ω^4 where $\omega = e^{2\pi i/5}$ is a primitive 5th root of unity. But then any $\sigma \in G = \mathrm{Aut}_K(L)$ is determined by where it sends ω, for which there are only 4 choices. So $\# G = 4$, and in fact $G \simeq \mathbb{Z}/4\mathbb{Z} \neq S_4$. //

So how do we figure out what the Galois group of an (irreducible) polynomial $f(x)$ is? We can start w/ small degrees.

Thm If $f(x)$ is an irreducible polynomial of degree 2, then its Galois group is $\mathbb{Z}/2\mathbb{Z}$.

Pf: This is the only transitive subgroup of $S_2 = \mathbb{Z}/2\mathbb{Z}$ ⊠

For degree 3, we will need an invariant of our $f(x)$. From now on let's assume that char $K \neq 2$ because that can cause some problems...

Def'n Let $f(x) \in K[x]$ be a polynomial and write $f(x) = \prod_{i=1}^{n} (x - u_i)$ in its splitting field L, so $u_1, u_2, \ldots, u_n$ are the roots of f.

Define $\Delta = \prod_{1 \le i < j \le n} (u_i - u_j) = (u_1 - u_2)(u_1 - u_3) \cdots (u_{n-1} - u_n)$.

The discriminant of $f(x)$ is $D = \Delta^2$.

Remark: Δ is the "Vandermonde determinant" evaluated at the roots of $f(x)$.

Remark: Notice that the discriminant of $f(x) = 0$ $\Leftrightarrow$ two of the roots of $f(x)$ coincide.

A priori, the discriminant D of $f(x)$ is just an element $D \in L$ in the splitting field, but in fact ...

Thm For any $\sigma \in \mathrm{Aut}_K(L)$, we have $\sigma(\Delta) = \Delta$ if σ is an even permutation and $\sigma(\Delta) = -\Delta$ if it's an odd permutation. Hence $\sigma(D) = D$ for all $\sigma \in \mathrm{Aut}_K(L)$.

Pf. Recall that a permutation is even if it is a product of an even number of transpositions, and the sign of a permutation σ is 1 if it is even and -1 if not. We proved last semester that the sign of σ can also be expressed as $(-1)^{\#\text{ inversions of } \sigma}$ where an inversion of $\sigma \in S_n$ is a $i < j$ such that $\sigma(i) > \sigma(j)$. This gives the fact that $\sigma(\Delta) = \Delta$ for even perms and $\sigma(\Delta) = -\Delta$ for odd ∎

Cor The discriminant D of $f(x)$ in fact belongs to the base field K,

Pf: By the Fund. Thm. of Galois Theory, or really just the fact that L/K is Galois, the subfield of elements fixed by all $\sigma \in \mathrm{Aut}_K(L)$ is K, hence $D \in K$. ∎

E.g. For $f(x)=ax^2+bx+c$ a quadratic, you may remember that the part b^2-4ac is called the discriminant in the quadratic equation, and now you know why.

Thm The extension $K(\Delta)$ is the field corresponding to the intersection of $G=\mathrm{Aut}_K(L)\subseteq S_n$ with the alternating group $A_n\subseteq S_n$ of even perm's.
In particular, $\Delta\in K \Leftrightarrow G\subseteq A_n$. Pf: Exercise for you using Fund. Thm. of Galois Theory. ∎

This lets us deal with cubic irreducible polynomials.

Thm Let $f(x)$ be a degree 3 irreducible polynomial. Then the Galois group of f is $A_3=\mathbb{Z}/3\mathbb{Z}$ if its discriminant D is a square in K, and is otherwise the full symmetric group S_3.
Pf: These are the transitive subgroups of S_3... ∎

Remark: A cubic $f(x)=x^3+bx^2+cx+d\in K[x]$ can be brought to the form x^3+px+q by replacing $f(x)$ with $f(x-b/3)$.

Thm For a cubic in form $f(x)=x^3+px+q$, its discriminant is equal to $D=-4p^3-27q^2$.
Pf: Long, "gruesome" computation... see book.

E.g. On HW#2 you had the poly. $f(x)=x^3-2\in\mathbb{Q}[x]$, whose discriminant $D=-27(2)^2=-108$ is not a square in $\mathbb{Q}$, hence its Galois group is the full S_3.

E.g. For $f(x)=x^3-3x+1\in\mathbb{Q}[x]$ (which is irreducible), its discriminant is $D=-4(-3)^3-27(1)^2=108-27=81$, which is a square in $\mathbb{Q}$, so its Galois group is only $A_3=\mathbb{Z}/3\mathbb{Z}$.

For quartics (degree 4), see the textbook. It just keeps getting more complicated!