Math 211 (Modern Algebra II), HW# 1,

Spring 2025; Instructor: Sam Hopkins; Due: Wednesday, January 29th

1. Let K be a field and L/K a finite extension. Recall $[L : K]$ denotes the degree of L over K. Prove the following:
   (a) $[L : K] = 1$ if and only if $L = K$.
   (b) If $[L : K]$ is a prime number, then there are no intermediate fields between K and L.
   (c) If $u \in L$ is an algebraic element of degree n over K, then n divides $[L : K]$.

2. Let K be a field. Recall that $K[x]$ denotes the polynomial ring, and $K(x)$ denotes the field of rational functions, both with coefficients in K. We have seen that a basis of $K[x]$ as a K-vector space is $\{x^j : j \geq 0\}$. Prove that the following is a K-basis of $K(x)$:

$$\{x^j : j \geq 0\} \cup \left\{ \frac{x^j}{P(x)^k} : k \geq 1,\, P(x) \in K[x] \text{ monic and irreducible, } 0 \leq j < \deg(P(x)) \right\}.$$

   Hint: remember the partial fraction decomposition of a rational function.

3. Let $f(x) = x^3 - 2x + 2 \in \mathbb{Q}[x]$, a polynomial which is irreducible over the rational numbers. (Look up "Eisenstein's criterion" if you want to see why it is irreducible.) In fact, $f(x)$ has a unique real root, call it $u \in \mathbb{R}$. Let $L = \mathbb{Q}(u)$. We have seen that $\{1, u, u^2\}$ is a $\mathbb{Q}$-basis of L.
   (a) Express $u^4 - 2u^3 + u^2 - 4 \in L$ as a $\mathbb{Q}$-linear combination of $\{1, u, u^2\}$.
   (b) Express $(u^2 - 3u + 1)^{-1} \in L$ as a $\mathbb{Q}$-linear combination of $\{1, u, u^2\}$.

   Hint: like we saw in class, use polynomial long division and the Euclidean gcd algorithm.

4. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find $[L : \mathbb{Q}]$ and find a $\mathbb{Q}$-basis of L.

5. Recall that a real number $c \in \mathbb{R}$ is called *constructible* if we can produce the point $(0, c) \in \mathbb{R}^2$ starting from the integer lattice $\mathbb{Z}^2 \subseteq \mathbb{R}^2$ and using a straightedge and compass. Prove that if $c \geq 0$ is constructible, then $\sqrt{c}$ is constructible.