Math 211 (Modern Algebra II), Midterm Exam,

Spring 2025; Instructor: Sam Hopkins; Taken on: Wednesday, February 26th

Each problem is worth 10 points, for a total of 50 points. You have 80 minutes to do the exam. Partial credit will be given generously, so write as much as you know for each problem.

1. Let $f(x) = x^2 - x - 1 \in \mathbb{Q}[x]$, a polynomial which is irreducible over $\mathbb{Q}$, and let u be the unique positive real root of $f(x)$. Consider $L = \mathbb{Q}(u)$ as extension of $K = \mathbb{Q}$.
   (a) What is the degree $[L : K]$?
   (b) Write a basis of L over K.
   (c) Express $u^2 + u + 1$ in your basis.
   (d) Express u^{-1} in your basis.
2. (a) Give a specific example of a field extension which is finitely generated but not algebraic. Explain why your example works.
   (b) Give a specific example of a finite extension of $\mathbb{Q}$ which is not Galois. Explain why your example works.
3. Give a specific example of a finite extension of fields L/K and a subgroup $H \subseteq \mathrm{Aut}_K(L)$ of the Galois group whose fixed field $H' = \{u \in L \colon \sigma(u) = u \text{ for all } \sigma \in H\}$ is neither K nor L. Explain why your example works.
4. Does the polynomial $f(x) = x^3 + 4x \in \mathbb{R}[x]$ split over the real numbers $\mathbb{R}$? Explain what this means and why or why not. If it does not split over $\mathbb{R}$, then what is the smallest extension of $\mathbb{R}$ where it does split? Show *how* it splits, either in $\mathbb{R}$ or in the extension.
5. An element $x \in K$ of a field K is called a *square* if $x = y^2$ for some $y \in K$. Let K be a finite field of characteristic 2. Prove that every element of K is a square. **Hint**: recall the Frobenius automorphism.