8/23 Fall 2021, Howard Math 273:
Combinatorics I (1st semester intro grad comb.)
Instructor: Sam Hopkins, samuelfhopkins@gmail.com
Website: samuelfhopkins.com/classes/273.html

Class info:
- Meets MWF ~~12-1~~ 12-1, online via Zoom (but also F2F in office!)
- Office hrs: TBD (email me to set up a meeting!)
- Text: R. Stanley's "Enumerative Combinatorics, Vol. 1" (linked to on course website) ← HW exercises from here!

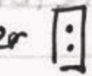
  Also closely follow notes of F. Ardila (link on website)
- Grading: There are 3 HW's (roughly: Oct., Nov., Dec.)
  Beyond that I expect you to show up to and participate in class (ask questions!)
- Disclaimer: This class is heavily based on a class I taught in Fall 2019 at U. of Minnesota
  In turn that class was based on class of V. Reiner.

What is this class about?

We want to count sets of discrete objects (subsets, multisets, partitions, graphs, etc.)
and more generally understand their structure (e.g., partial order structure).

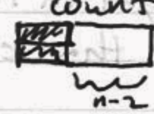
What is a "good" answer to a counting problem?
It depends! (for instance, on what we want to do w/ answer...)
Rather than try to formalize a notion of "good answer", let's explain what answers can look like in an example...

Ex: Let a_n = # tilings of $2 \times n$ rectangles by 2×1 dominoes

either  or 

n	a_n	rectangle	tilings
0	1	1	1
1	1		
2	2		
3	3		
4	5		 a_3, a_2

① Recurrence! $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$, w/ $a_0 = a_1 = 1$

counts  $n-1$ counts  $n-2$

This is the same recurrence as the Fibonacci numbers:

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

Usually F_i are indexed w/ $F_0 = 0$ and $F_1 = 1$

n	a_n	F_n
0	1	0
1	1	1
2	2	1
3	3	2
4	5	3

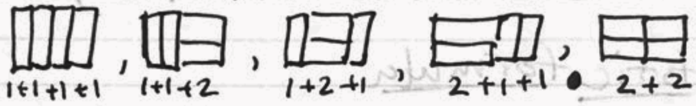
so $a_n = F_{n+1}$, i.e., we are just considering Fibonacci numbers with slightly different indexing.

This recurrence allows us to compute many values of a_n, but it could take a while... and don't know growth rate!

② Explicit formula as a summation.

Observe $a_n = \#\{\text{sequences of 1's and 2's summing to } n\}$

e.g., $n=4$



$1+1+1+1$, $1+1+2$, $1+2+1$, $2+1+1$, $2+2$

8/25 $\Rightarrow a_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \#\{\text{seq's of } k \text{ 2's and } n-2k \text{ 1's}\}$

$\left(\text{e.g. } 1+2+1+1+2+1+2 = 10 \right)$ $k=3$

$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{(n-2k)+k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$ ← we'll discuss binomial coeff's, a.k.a. "a choose b", more later...

e.g. $a_4 = \binom{4}{0} + \binom{4-1}{1} + \binom{4-2}{2} = \binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 1+3+1 = 5$

This formula is "explicit", but still not so fast to compute + still does not give sense of growth of seq. a_n ...

Also, existence of this formula doesn't mean there isn't a better one!

e.g. $\#\text{subsets of } \{1,2,\dots,n\} = \sum_{k=0}^{n} \binom{n}{k}$ ← "explicit" formula

$= 2^n$ ← better formula!

③ Explicit formula w/ exponentiation.

The recurrence relation $a_n = a_{n-1} + a_{n-2}$ (w/ initial conditions $a_0 = a_1 = 1$)

implies that $a_n = \frac{1}{\sqrt{5}} \left(\Big(\underbrace{\frac{1+\sqrt{5}}{2}}_{\varphi}\Big)^{n+1} - \Big(\underbrace{\frac{1-\sqrt{5}}{2}}_{\psi}\Big)^{n+1} \right)$

(You may have seen this before e.g. in a linear algebra course.)

We'll explain why this formula holds very soon.
This formula is very explicit, and it does show growth rate of a_n, but it does have its own drawbacks: why is it even an <u>integer</u>?

④ <u>Asymptotic formula</u>

Can compute $\begin{cases} \varphi \approx 1.618\ldots & (\text{"golden ratio"}) \\ \psi \approx -0.618\ldots \end{cases} \Rightarrow |\varphi| > |\psi|$

which means $a_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$ as $n \to \infty$. This gives precise understanding of growth rate; e.g., # of digits of a_n is $\log_{10}(a_n) \approx (n+1)\underbrace{\log_{10}(\varphi)}_{\approx 0.2} - \underbrace{\log_{10}(\sqrt{5})}_{\text{very small}}$

⑤ <u>(Ordinary) generating function for a_n</u>

$$A(x) \overset{\text{DEF'N}}{:=} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$(= 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots)$$
$$= \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$$

the ring of <u>formal power series</u> in x, with coefficients in $\mathbb{C}$
(We'll give a formal definition of this algebraic structure in a short while...)

Not so clear at first why you'd ever consider $A(x)$, but we'll see that generating functions are <u>very powerful</u>, e.g., we can derive everything we saw about (a_n) from $A(x)$.

<u>Claim</u>: $A(x) = \dfrac{1}{1-x-x^2}$

Pf: Recall recurrence $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ (and $a_0 = a_1 = 1$)

Multiply by x^n and sum over all $n \geq 2$ to get:

$$\sum_{n\geq 2} a_n x^n = \sum_{n\geq 2} a_{n-1} x^n + \sum_{n \geq 2} a_{n-2} x^n$$

$$A(x) - a_0 x^0 - a_1 x^1 = x\left(\sum_{m\geq 1} a_m x^m\right) + x^2\left(\sum_{m\geq 0} a_m x^m\right)$$

$$A(x) - 1 - x = x(A(x) - a_0 x^0) + x^2(A(x))$$

$$(1 - x - x^2)\cdot A(x) = x + 1 - x = 1$$

$$\Rightarrow A(x) = \frac{1}{1-x-x^2}. \quad \blacksquare$$

8/28 What good is knowing $A(x) = \frac{1}{1-(x+x^2)}$? Plenty!

Let's extract coefficients of $A(x)$ in various ways...

(a) $A(x) = \frac{1}{1-(x+x^2)} = 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \ldots$ using $\frac{1}{1-r} = 1 + r + r^2 + \ldots$ (geometric series)

i.e., $\sum_{n\geq 0} a_n x^n = \sum_{d \geq 0} (x+x^2)^d \overset{\text{binomial thm.}}{=} \sum_{d\geq 0}\left(\sum_{k=0}^{d} \binom{d}{k} \underbrace{(x^2)^k x^{d-k}}_{x^{d+k}}\right)$

$$= \sum_{n\geq 0} x^n \left(\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\right) \quad \begin{array}{l} n = d+k \\ d = n-k \end{array}$$

$\Rightarrow a_n = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k}$, our first explicit formula from before.

(b) $$A(x) = \frac{1}{1-x-x^2} = \frac{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)}{1 - \frac{1+\sqrt{5}}{2}x} + \frac{\frac{-1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)}{1 - \frac{1-\sqrt{5}}{2}x}$$

How to see this?

Recall partial fractions:

$$\frac{1}{ax^2+bx+c} = \frac{1}{a(x-r_1)(x-r_2)} = \frac{A}{x-r_1} + \frac{B}{x-r_2} = \frac{-A/r_1}{1-\frac{x}{r_1}} + \frac{-B/r_2}{1-\frac{x}{r_2}}$$

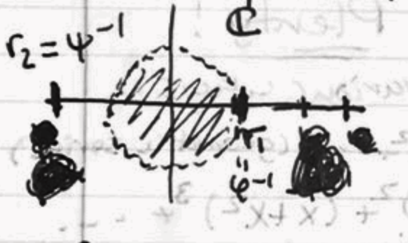
Here $r_1 = \left(\frac{1+\sqrt{5}}{2}\right)^{-1} = \varphi^{-1}$, $r_2 = \left(\frac{1-\sqrt{5}}{2}\right)^{-1} = \psi^{-1}$

$$\Rightarrow A(x) = \frac{1}{\sqrt{5}} \sum_{n\geq 0} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} x^n - \frac{1}{\sqrt{5}} \sum_{n\geq 0} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} x^n$$

(again using geo. series)

$$\Rightarrow a_n = \frac{1}{\sqrt{5}} \Bigg(\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\varphi}^{n+1} - \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\psi}^{n+1} \Bigg),$$ our 2^{nd} explicit formula from before ...

(c) The asymptotic $a_n \approx C \overbrace{\left(\frac{1+\sqrt{5}}{2}\right)}^{r_1^{-1}}{}^{n}$ for some constant C corresponds to the fact that r_1^{-1} is the reciprocal of the pole of $A(x) = \frac{1}{1-x-x^2} = \frac{1}{(x-r_1)(x-r_2)}$ nearest the origin of $\mathbb{C}$



This is just the tip of the rich interplay between thinking of $A(x)$ as a formal power series, and as an (analytic) function of a complex variable

(For more, see H. Wilf's "generatingfunctionology," linked to on website)

8/30 The fast way to get $A(x) = \frac{1}{1-x-x^2}$ is via Pólya's "picture-writing":

$$\frac{1}{1-(\underset{P}{▯} + \underset{Q}{⊟})} = \underset{n=0}{1} + (\underset{n=1}{▯} + \underset{n=2}{⊟})^1 + (▯ + ⊟)^2 + (▯ + ⊟)^3 + \cdots$$

$\mathbb{C}[[P,Q]]$; $P = x^1$, $Q = x^2$

$(▯+⊟)^2 = (▯+⊟)(▯+⊟) = \underset{n=2}{▯▯} + \underset{n=3}{▯⊟} + \underset{n=3}{⊟▯} + \underset{n=4}{⊟⊟}$

$(▯+⊟)^3 = (▯+⊟)(▯+⊟)(▯+⊟)$; $(+\cdots+ \underset{n=4}{⊟▯▯} + \cdots)$

$▯ \mapsto x^1$, $⊟ \mapsto x^2$

$$\mathbb{C}[[x]] \ni A(x) = \frac{1}{1-(x+x^2)} \quad \left(= 1 + (x+x^2)^1 + (x+x^2)^2 + \cdots \right)$$

The generating function can often be refined to keep track of additional statistics on our combinatorial objects.

Say we want to compute

$a_{m,n} = \#\{\text{domino tilings of } 2\times n \text{ rect. w/ } m \text{ vertical tiles}\}$

From "picture-writing" we get

"weight" vertical tile by formal parameter v

$$\left[\frac{1}{(1-(\underset{P}{\square}+\underset{Q}{\boxminus}))}\right]_{\substack{P=vx\\ Q=x^2}} = \sum_{n,m\geq 0} a_{m,n}\, x^n v^m \in \mathbb{C}[[x,v]]$$

$$= \frac{1}{1-vx-x^2}$$

This (two variable) g.f. is useful for e.g. computing (asymptotically) the expected number of vertical tiles in a random tiling:

$$\sum_{n\geq 0}\left(\sum_{m\geq 0} a_{m,n}\cdot m\right)x^n = \left[\partial/\partial v \sum_{n,m\geq 0} a_{m,n}\, x^n v^m\right]_{v=1}$$

$$= \left[\partial/\partial v\ \frac{1}{1-vx-x^2}\right]_{v=1}$$

$$= \left[\frac{x}{(1-vx-x^2)^2}\right]_{v=1} = \frac{x}{(1-x-x^2)^2}$$

Via partial fractions $\left(\frac{x}{(1-x-x^2)^2} = \frac{A_1x+B_1}{(x-r_1)^2} + \frac{A_2x+B_2}{(x-r_2)^2} + \frac{C}{(x-r_1)} + \frac{D}{(x-r_2)}\right)$

Can use above formula to show

$$\text{total \# of vertical tiles in all tilings of } 2\times n \text{ rect.} = \sum_{m\geq 0} a_{m,n}\cdot m \approx \frac{n}{5}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$$

Recall $a_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \Rightarrow \sum_{m\geq 0} a_{m,n}\cdot m \approx \frac{n}{\sqrt{5}}\cdot a_n$.

Thus, the expected # of vertical tiles is $\approx \frac{n}{\sqrt{5}}$, i.e., out of the n tiles in a tiling of $2\times n$ rectangle, about $\frac{1}{\sqrt{5}} \approx 44.7\%$ of them will be vertical. //

9/1

<u>The ring of formal power series</u> $R[[x]]$

[where $R = \mathbb{C}$ or $\mathbb{R}$ or $\underbrace{\mathbb{C}[v]}_{\text{polynomial ring}}$ or any commutative ring w/ 1)

<u>DEF'N</u> $R[[x]] := \{ a_0 + a_1 x + a_2 x^2 + \cdots \overset{A(x)}{=} \sum_{n=0}^{\infty} a_n x^n$ w/ $a_0, a_1, \ldots \in R\}$

is a <u>commutative ring</u> w/ <u>coefficientwise addition</u> $: A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n$

and <u>multiplication via convolution</u>:

$$C(x) := A(x) \cdot B(x) = \sum_{n \geq 0} c_n x^n \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$$

So its zero is $0 = 0 + 0x + 0x^2 + \cdots$

and its one is $1 = 1 + 0x + 0x^2 + \cdots$

<u>Prop.</u> $A(x) = \sum_{n=0}^{\infty} a_n x^n \in R[[x]]$ is a <u>unit</u> (i.e., $\exists B(x)$ w/ $1 = A(x)B(x)$)

$\Leftrightarrow$ a_0 is a unit of R (i.e., $\exists\, b_0 \in R$ w/ $1 = a_0 b_0$).

<u>E.g.</u>: By this criterion, $(1 - x - x^2) \in \mathbb{C}[[x]]$ is a unit,

so $\exists B(x)$ w/ $B(x) \cdot (1 - x - x^2) = 1$, i.e. $B(x) = \frac{1}{1 - x - x^2} = 1 + x + 2x^2 + 3x^3 + \cdots$

<u>Proof</u>: $\underset{1 + 0x + 0x^2 + \cdots}{\overset{1}{\shortparallel}} = A(x) B(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 + \cdots$

$\Leftrightarrow a_0 b_0 = 1$, so we need a_0 to be a unit in R (i.e., $b_0 = a_0^{-1}$ in R)

and then we have $a_0 b_1 + a_1 b_0 = 0$ meaning $b_1 = \frac{-a_1 b_0}{a_0}$ allowed to divide by a_0 since it's a unit

$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0$ meaning $b_2 = \frac{-(a_1 b_1 + a_2 b_0)}{a_0}$ all terms already defined

$\cdots$ can recursively define all b_i in a unique way. $\blacksquare$

9/3

<u>DEF'N</u> A sequence $A_0(x), A_1(x), A_2(x), \ldots$ in $R[[x]]$ <u>converges</u> (i.e., $\exists A(x) = \lim_{j\to\infty} A_j(x)$) if $\forall n \geq 0$, the <u>coefficient of x^n in $A_j(x)$</u> stabilizes for $j >> 0$.

call this $[x^n]A_j(x)$ I.e., $\forall n \geq 0$, $\exists N \geq 0$ and $a_n \in R$ s.t. $[x^n]A_j(x) = a_n \ \forall j \geq N$.

<u>E.g.</u> $A(x) = \frac{1}{1-(x+x^2)} = \underbrace{\underbrace{\underbrace{1}_{A_0(x)} + (x+x^2)}_{A_1(x)} + (x+x^2)^2}_{A_2(x)} + (x+x^2)^3 + \cdots$ ...

converges in $\mathbb{C}[[x]]$; e.g., $[x^3]A(x) = [x^3]A_3(x) = [x^3]A_4(x) = \cdots = a_3 = 3$.

<u>E.g.</u> $e^{x+1} := 1 + \frac{(x+1)}{1!} + \frac{(x+1)^2}{2!} + \cdots$ does <u>not</u> converge in $\mathbb{C}[[x]]$ (even if it makes sense analytically)

but $e^x := 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ does converge.

Alternatively, $\{A_j(x)\}_{j=0,1,\ldots}$ converges in $R[[x]]$ if $\lim_{j\to\infty} \operatorname{mindeg}(A_j(x) - A_{j-1}(x)) = \infty$,

where <u>DEF'N</u> $\operatorname{mindeg} A(x) :=$ smallest d w/ $a_d \neq 0$ (or ∞ if no such d)
$\sum_{n\geq 0} a_n x^n$

<u>e.g.</u> $A(x) = \frac{1}{1-(x+x^2)}$, then $A_j(x) - A_{j-1}(x) = (x+x^2)^j$ and $\operatorname{mindeg}(x+x^2)^j = j \to \infty$ as $j \to \infty$

<u>Remark</u> with <u>metric</u> $d(A(x), B(x)) = \frac{1}{2^{\operatorname{mindeg}(A(x)-B(x))}}$,
$R[[x]]$ is a "complete topological ring"
(so basic stuff you know about topologies, convergence, limits, etc. works for $R[[x]]$.)
In fact, it is the completion of polynomial ring $R[x]$.

Cor $\sum_{j=0}^{\infty} B_j(x) = \underbrace{\underbrace{\underbrace{B_0(x)}_{A_0(x)} + B_1(x)}_{A_1(x)} + B_2(x)}_{A_2(x)} + \cdots$ converges in $R[[x]]$ $\Leftrightarrow$ mindeg $B_j(x) \to \infty$ as $j \to \infty$

$\parallel$

$\lim_{n\to\infty} A_n(x)$

w/ $B_j = A_j - A_{j-1}$

Cor Infinite products of the form

$\prod_{j=1}^{\infty} (1 + B_j(x))$ w/ mindeg $B_j \geq 1 \;\forall j$

converges in $R[[x]]$ $\Leftrightarrow$ mindeg $B_j \to \infty$ as $j \to \infty$

$= \lim_{n\to\infty} A_n(x)$ where $A_0 = 1$, $A_1 = (1 + B_1(x))$,

$A_2 = (1+B_1(x))(1+B_2(x)), \ldots, A_j = A_{j-1}(1+B_j)$

E.g. $\underbrace{\prod_{n=1}^{\infty} (1 + \frac{1}{2^n} x)}_{\text{"}A(x)\text{"}}$ does not converge in $\mathbb{C}[[x]]$

(even if it does make sense to think of $A(x)$ as a function of $x \in \mathbb{C}$ for $|x|$ small...)

E.g. $\prod_{n=1}^{\infty} (1 + x^n)$ converges in $R[[x]]$

$\parallel$

$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$

$\parallel$

$\sum_{n\geq 0} a_n x^n$ Q: What are these coefficients a_n? What is their combinatorial significance?

A: We will see next class, when we discuss integer partitions!