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Partitions and their generating functions

DEF'N An (integer) partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of n is a • weakly decreasing, eventually zero, … $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \dots \ge 0$ sequence of nonnegative integers with $\lambda_1 + \lambda_2 + \lambda_3 + \dots = n$. ← 'size' of λ

We write $\lambda \vdash n$ ('λ \vdash n') and $|\lambda| = n$.

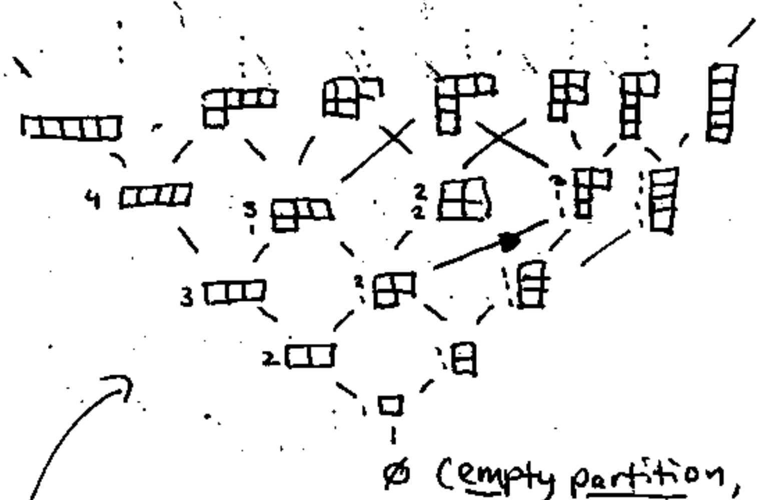
e.g. $\lambda = (5,5,3,1,0,0,\dots) = (5,5,3,1,0) = (5,5,3,1) \vdash 14 = 5+5+3+1$

Its length $\ell(\lambda) := \#\{i : \lambda_i > 0\}$ = # of nonzero parts λ_i.

Its Young diagram is a left + top justified array of boxes, with λ_i boxes in the i^{th} row from the top:

e.g., $\lambda = (5,5,3,1) \leftrightarrow$ 

Let $p(n)$:= # of partitions $\lambda \vdash n$



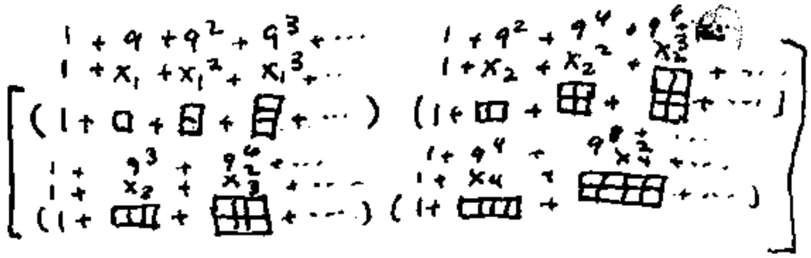
∅ (empty partition, unique partition of 0)

p(n)	n
…	…
7	5
5	4
3	3
2	2
1	1
1	0

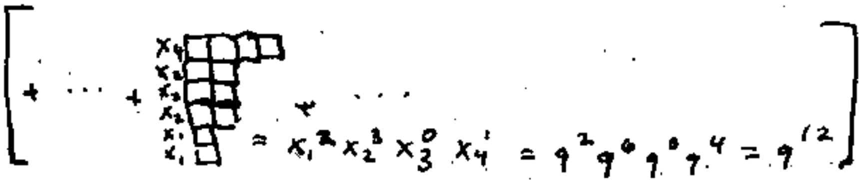
$\mathbb{Y}$ = Young's lattice, the poset of all partitions, ordered by containment of Young diagrams

$\overset{\lambda}{\underset{\mu}{\vdots}}$ means λ obtained from μ by adding one box

$$\sum_{n\geq 0} \underset{\underset{\mathbb{C}[[q]]}{\Uparrow}}{p(n)}\, q^n = \sum_{\substack{\text{all partitions}\\ \lambda}} q^{|\lambda|} =$$



$$\cdots =$$



$$= x_1^2 x_2^3 x_3^0 x_4^1 = q^2 q^6 q^0 q^4 = q^{12}$$

$$x_i \mapsto q^i = (1+q+q^2+\cdots)(1+q^2+q^4+\cdots)(1+q^3+q^6+\cdots)$$

$$= \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdot \cdots$$ a convergent product

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$ ← g.f. for all partitions, as a product formula!

Cultural asides:

Thm (Euler's pentagonal number theorem) ← "pentagonal numbers"

$$\prod_{n=1}^{\infty} (1-q^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(q^{k(3k+1)/2} + q^{k(3k-1)/2}\right)$$

$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots$$

Pf: We will give a bijective proof later in the course. ☒

Since $\sum p(n) q^n = \left(\prod_n (1-q^n)\right)^{-1}$, Pent. # Thm implies a recurrence for $p(n) = \sum_{k=-\infty, k\neq 0}^{\infty} (-1)^k p(n-k(3k-1)/2)$

$$= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots$$

Thm (Hardy + Ramanujan) $p(n) \approx \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$

Pf: "Circle method," careful analysis of the singularities of $\prod \frac{1}{1-q^n}$ thought of as an analytic function. Like what we saw with rational g.f. of Fibonacci #'s, but much more hardcore complex analysis… ☒

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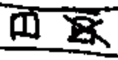
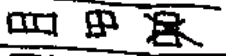
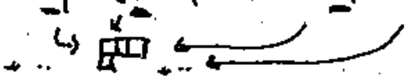
G.f.'s for restricted classes of partitions:

Let $d(n)$:= # of partitions of n into distinct parts.

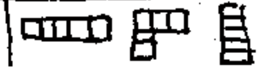
n	$d(n)$	
0	1	∅
1	1	
2	1	
3	2	
4	2	
5	3	

$$D(q) := \sum_{n\geq 0} d(n) q^n = (1+q)(1+q^2)(1+q^3)(1+q^4)\cdots$$



$$= \prod_{j=1}^{\infty} (1+q^j)$$

Let $o(n)$:= # partitions of n into odd parts.

n	$o(n)$	
0	1	∅
1	1	
2	1	
3	2	
4	2	
5	3	

Looks like possibly $o(n) = d(n)$, but how to show it?

G.f's!

$$O(q) := \sum_{n\geq 0} o(n) q^n = (1+q+q^2+\cdots)(1+q^3+q^6+\cdots)(1+q^5+q^{10}+\cdots)$$

$$= \frac{1}{1-q}\cdot\frac{1}{1-q^3}\cdot\frac{1}{1-q^5}\cdots = \prod_{j=1}^{\infty}\frac{1}{1-q^{2j+1}} \overset{?!}{=} \prod_{j=1}^{\infty}(1+q^j)$$

How to show $D(q) = O(q)$?

Well, $D(q) = (1+q)(1+q^2)(1+q^3)\cdots$

$$= \frac{(1-q^2)}{(1-q)}\frac{(1-(q^2)^2)}{(1-q^2)}\frac{(1-(q^3)^2)}{(1-q^3)}\cdots$$

Recall: $(1+x)(1-x) = (1-x^2)$

$$= \frac{\cancel{(1-q^2)}(1-\cancel{q^4})(1-\cancel{q^6})(1-\cancel{q^8})\cdots}{(1-q)\cancel{(1-q^2)}(1-q^3)\cdot\cancel{(1-q^4)}\cdots}$$

$$= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots} = O(q) \quad !$$

Was that manipulation ok? Yes! Thinking slightly differently...

Let $R(q) := (1-q)(1-q^3)(1-q^5)\cdots = \frac{1}{O(q)} \in \mathbb{C}[[q]]$

Want to show "$1 \overset{?}{=} D(q)R(q)$ in $\mathbb{C}[[q]]$

$1+0\cdot q+0\cdot q^2+\cdots$

$$\shortparallel$$

$$((1+q)(1+q^2)(1+q^3)\cdots)((1-q)(1-q^3)(1-q^5)\cdots)$$

$$= \underbrace{(1+q)(1-q)}_{(1-q^2)}((1+q^2)(1+q^3)\cdots)((1-q^3)(1-q^5)\cdots)$$

$$= (1-q^4)((\underbrace{1+q^3)(1+q^4)\cdots)((1-q^3)}(1-q^5)\cdots)$$

$$= \underbrace{(1-q^4)(1-q^6)((1+q^4)}_{\cdots}(1+q^5)\cdots)((1-q^5)(1-q^7)\cdots)$$

$$= 1 + 0\cdot q + 0\cdot q^2 + 0\cdot q^3 + \cdots$$

$= \cdots$ et cetera

hinted at by this manipulation

Note: $\exists$ bijective proof that $d(n) = o(n)$ as well

(See Stanley Prop. 1.8.5 for 2 such proofs...) Basic idea is binary expansion:

e.g. $\underset{o(n)}{\lambda} = (9^5, 5^{12}, 3^2, 1^3) = (9^{2^0+2^2}, 5^{2^2+2^3}, 3^{2^1}, 1^{2^0+2^1})$

$\overset{n}{(9,9,9,9,9,\underbrace{5,5,\ldots,5}_{12},3,3,1,1,1)}$

$n=114$

$$\longleftrightarrow \mu = (9\cdot 2^0, 9\cdot 2^2, 5\cdot 2^2, 5\cdot 2^3, 3\cdot 2^1, 1\cdot 2^0, 1\cdot 2^1)$$

$$= (9, 36, 20, 40, 6, 1, 2)$$

(reorder)

$$= (40, 36, 20, 9, 6, 2, 1) \in d(n)$$

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Some useful formal power series

Let's define and study some specific elements of $\mathbb{C}[[x]]$:

DEF'N $e^x := \sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\log(1+x) := \sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\forall \lambda \in \mathbb{C}$, $(1+x)^\lambda := \sum_{k\geq 0} \binom{\lambda}{k} x^k$

where $\binom{\lambda}{k} := \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-(k-1))}{k!} \in \mathbb{C}$ ← "generalized binomial coefficient"

(just like for $n \in \mathbb{N}$, $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-(k-1))}{k!}$)

These formal power series satisfy the properties you'd expect:

E.g. ① $(1+x)^\lambda (1+x)^\mu = (1+x)^{\lambda+\mu} \in \mathbb{C}[[x]]$ $\forall \lambda, \mu \in \mathbb{C}$

② $e^x e^y = e^{x+y} \in \mathbb{C}[[x,y]]$

③ $\underbrace{e^{\log(1+x)}} = 1 + x$, etc...

defined to be $= 1 + \log(1+x) + \frac{\log(1+x)^2}{2!} + \cdots$

$= 1 + (x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots) + \frac{(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots)^2}{2} + \cdots$

Why does this even converge in $\mathbb{C}[[x]]$?

Prop. If $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, and $b_0 = 0$, then $A(B(x)) := \sum_{n\geq 0} a_n (B(x))^n$ converges in $\mathbb{C}[[x]]$. ⌟

How to justify ①, ②, ③, etc...? Could do a tedious manipulation of coefficients, but instead, since e^x, $\log(1+x)$, $(1+x)^\lambda$ are also analytic functions we are familiar with (whose power series expansions are as above), we can use a trick from complex analysis...

↙ standard fact from complex analysis, true under weaker hypotheses too...

<u>Thm</u> If $f(z)=\sum_{n\geq 0} a_n z^n$ is <u>analytic</u> for $|z|<R$ for some $R>0$, and f vanishes on $|z|<R$, then $a_0=a_1=a_2=\cdots=0$.

④ for $n\in\mathbb{N}$, $(1+x)^n=\sum_{k\geq 0}\binom{n}{k}x^k=\sum_{k=0}^{n}\binom{n}{k}x^k$,

but also $\frac{1}{(1-x)^n}=(1+(-x))^{-n}=\sum_{k\geq 0}\frac{(-n)(-n-1)(-n-2)\cdots(-n-(k-1))}{k!}(-x)^k$

onion ← " n terms in product → sesame asiago

$$\underbrace{(1+x+\boxed{x^2}+\cdots)(1+\boxed{x}+x^2+\cdots)\cdots(1+x+\boxed{x^2}+\cdots)}_{\ast\ast|\ast||\cdots|\ast|\ast\ast} = \sum_{k\geq 0}\frac{n(n+1)(n+2)\cdots(n+k-1)}{k!}x^k$$

e.g. n=18 flavors of bagels

$$=\sum_{k\geq 0}\binom{n+k-1}{k}x^k$$

<u>Pf</u> <u>that</u> $\left(\!\binom{n}{k}\!\right)=\binom{n+k-1}{k}$: ('n <u>multichoose</u> k') $\left(\!\binom{n}{k}\!\right) := \#$ k element multisets w/ entries in $\{1,2,\ldots,n\}$

'Stars and bars'

Stars indicate how many of each element is chosen, bars separate bins for each element

e.g. $\{1,1,2,4,4,4,7\}$ is a 7-element multisubset of $\{1,2,\ldots,9\}$

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⑤ $\underset{(1+(-4x))^{-1}}{\frac{1}{1-4x}}=\sum_{k\geq 0}\binom{-1}{k}(-4x)^k=\sum_{k\geq 0}\underbrace{\binom{1+k-1}{k}}_{=\binom{k}{k}=1}4^kx^k=\sum_{k\geq 0}4^kx^k$ ✓

but also $\frac{1}{(1-4x)^2}=\sum_{k\geq 0}\underbrace{\binom{2+k-1}{k}}_{\binom{k+1}{k}=k+1}4^kx^k=\sum_{k\geq 0}(k+1)4^kx^k$

$\frac{1}{(1-4x)^3}=\sum_{k\geq 0}4^kx^k\cdot\binom{k+2}{2}$, $\frac{1}{(1-4x)^4}=\cdots$ etc.

<u>useful</u> for extracting coefficients of a rational function after performing the <u>Partial</u> <u>fraction</u> <u>expansion</u>...

⑥ $$\frac{1}{\sqrt{1-4x}} = (1-4x)^{-1/2} = \sum_{k\geq 0} \binom{-\frac{1}{2}}{k} (-4)^k x^k$$

$$= \sum_{k\geq 0}\left(\frac{(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})\cdot\ (\frac{-(2k-1)}{2})}{k!}(-4)^k\right)x^k$$

this F.P.S. is algebraic but not rational

$$= \sum_{k\geq 0}\left(\frac{\overset{2^k}{\cancel{4^k}}(1)(3)(5)\cdots(2k-1)}{\cancel{2^k}\ k!}\right)x^k$$

$$= \sum_{k\geq 0}\left(\frac{\cancel{2^k}(1)(3)(5)\cdots(2k-1)}{k!}\ \frac{(2)(4)(6)\cdots(2k)}{\cancel{2^k}\ k!}\right)x^k$$

$$= \sum_{k\geq 0}\frac{(2k)!}{k!\cdot k!}x^k = \sum_{k\geq 0}\binom{2k}{k}x^k$$

central binomial coefficients... interesting...

Another tool from calculus that's useful for $R[[X]]$

DEF'N For $A(x) = \sum_{n\geq 0} a_n x^n \in R[[x]]$, we define the formal derivative $A'(x) := \sum_{n\geq 1} \underbrace{n\cdot a_n}_{\underbrace{a_n + a_n + \cdots a_n}_{n \text{ times}}} x^{n-1} \in R[[x]]$

The derivative satisfies the usual rules from calculus:

- $(A(x) + B(x))' = A'(x) + B'(x)$
- $(AB)' = A'\cdot B + B'\cdot A$
- $\left(\frac{1}{A}\right)' = \frac{-A'}{A^2}$
- $(A(B(x)))' = A'(B(x))\cdot B'(x)$
- etc...

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Quick review of binomial (and multinomial) coefficients

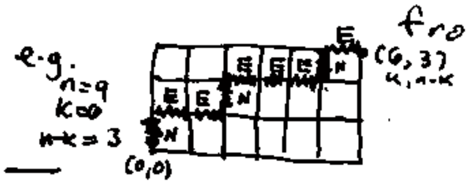
The Binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ has several (easy) interpretations

$= \#$ words with k 1's, $(n-k)$ 2's,

e.g. $\binom{4}{2} = 6 \longrightarrow$

$= \# \{1122, 1212, 1221, 2112, 2121, 2211\}$ i.e., rearrangements of $\underbrace{11\cdots1}_{k}\ \underbrace{22\cdots2}_{n-k}$

$= \#$ lattice paths in $\mathbb{Z}^2$ taking $\overset{E}{\text{east}}$ or $\overset{N}{\text{north}}$ steps, from $(0,0)$ to $(k, n-k)$



$\longleftrightarrow$
1 2 3 4 5 6 7 8 9
N E E N E E E N E $\longleftarrow \binom{9}{6}$
(could be 1's/2's instead of E's/N's)

$= \#$ subsets of $[n] := \{1, 2, \ldots, n\}$ of size k

e.g.
1 2 3 4 5 6 7 8 9
N E E N E E E N E $\longleftrightarrow \{2,3,5,6,7,9\} \subseteq [9]$
(position of E's in word)

Of course, we have

Thm (Binomial Theorem) $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Lemma (Pascal's Identity)
$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Pf (bijection):
$\begin{matrix} A \subseteq [n] \\ \#A = k \end{matrix} \mapsto \begin{cases} A & \text{if } n \notin A \subseteq [n-1] \\ A - \{n\} & \text{if } n \in A. \end{cases}$ ∎

$\Rightarrow$ Pascal's triangle

n=0 1
n=1 1 1
n=2 1 2 1
⋮
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1 . . .

Multinomials

How many rearrangements (anagrams) of BANANAS?

i.e., of 3A's, 1B, 2N's, 1S? (equiv. of $\overset{1}{A}\overset{2}{A}\overset{3}{A}\ \overset{4}{B}\overset{5}{N}\overset{6}{N}\ \overset{7}{S}$)

$\exists$ a <u>transitive</u> <u>action</u> of the <u>symmetric</u> <u>group</u> $\mathfrak{S}_7$ of <u>permutations</u> of $[7]$ on the rearrangements

e.g., perm. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 3 & 6 & 7 & 5 \end{pmatrix}$ sends $\overset{1}{A}\overset{2}{A}\overset{3}{A}\overset{4}{B}\overset{5}{N}\overset{6}{N}\overset{7}{S} \xrightarrow{\sigma} \overset{2}{A}\overset{1}{A}\overset{4}{B}\overset{3}{A}\overset{6}{N}\overset{7}{S}\overset{5}{N}$

The <u>Stabilizer</u> of this action is $\underset{\mathfrak{S}_{\{1,2,3\}}}{\underset{\shortparallel}{\mathfrak{S}_3}} \times \underset{\mathfrak{S}_{\{4\}}}{\underset{\shortparallel}{\mathfrak{S}_1}} \times \underset{\mathfrak{S}_{\{5,6\}}}{\underset{\shortparallel}{\mathfrak{S}_2}} \times \underset{\mathfrak{S}_{\{7\}}}{\underset{\shortparallel}{\mathfrak{S}_1}} \subseteq \mathfrak{S}_7$

So by <u>Orbit-Stabilizer Thm</u>, $\#\text{rearrangements} = \underset{\text{orbit}}{\text{size of}} = \dfrac{\#\mathfrak{S}_7}{\#\mathfrak{S}_3 \times \mathfrak{S}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_1}$

$$= \frac{7!}{3!\,1!\,2!\,1!} = \binom{7}{3,1,2,1}$$

The <u>Multinomial Coefficient</u> $\dbinom{n}{k_1, k_2, \dots, k_m} = \dfrac{n!}{k_1!\,k_2! \cdots k_m!}$ for $n = k_1 + k_2 + \dots + k_m$

$= \#$ words w/ k_1 1's, k_2 2's, …, k_m m's, i.e., rearrang's of $\underbrace{11\cdots1}_{k_1}\underbrace{22\cdots2}_{k_2}\cdots\underbrace{mm\cdots m}_{k_m}$

$= \#$ lattice paths in $\mathbb{Z}^m$ taking steps $\underset{\shortparallel}{\overset{(1,0,0,\dots,0)}{}}\vec{e}_1, \overset{(0,1,0,\dots,0)}{\vec{e}_2}, \dots, \overset{(0,0,\dots,1)}{\vec{e}_m}$
from $0 = (0, \dots, 0)$ to $(k_1, k_2, \dots, k_m)$
(same correspondence between words + walks as w/ binomials)

$= \#$ <u>chains</u> $\phi = S_0 \subset S_{k_1} \subset S_{k_1+k_2} \subset \cdots \subset S_{n=k_1+\dots+k_m} = [n]$
of subsets of $[n] = \{1, 2, \dots, n\}$ for which
$\# S_i = i \quad \forall\, i = 0, k_1, k_1+k_2, \dots, n$

<u>e.g.</u> for $\binom{7}{3,1,2,1}$ have $2131314 \Leftrightarrow \phi \subset \{2,4,6\} \subset \{1,2,4,6\} \subset \{1,2,3,4,5,6\} \subset [7]$

<u>Note</u> $\binom{n}{k} = \binom{n}{n, n-k}$ in multinomial notation

<u>Also</u> <u>note</u>: $\dbinom{n}{k_1, k_2, \dots, k_m} = \dbinom{n}{k_1} \cdot \dbinom{n-k_1}{k_2} \cdot \dbinom{n-(k_1+k_2)}{k_3} \cdots \dbinom{n-(k_1+\dots+k_{m-2})}{k_{m-1}} \dbinom{k_m}{k_m}$

And…
<u>Multinomial Theorem</u>: $$(x_1 + x_2 + \dots + x_m)^n = \sum_{\substack{(k_1, k_2, \dots, k_m) \\ k_1+k_2+\dots+k_m = n}} \binom{n}{k_1, k_2, \dots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

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