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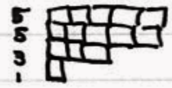
Partitions and their generating functions

DEF'N An (integer) partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ of n
is a • weakly decreasing,
eventually zero, $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \dots \ge 0$
sequence of nonnegative integers
with $\lambda_1 + \lambda_2 + \lambda_3 + \dots = n$. 'size' of λ
We write $\lambda \vdash n$ ('λ \vdash n') and $|\lambda| = n$.

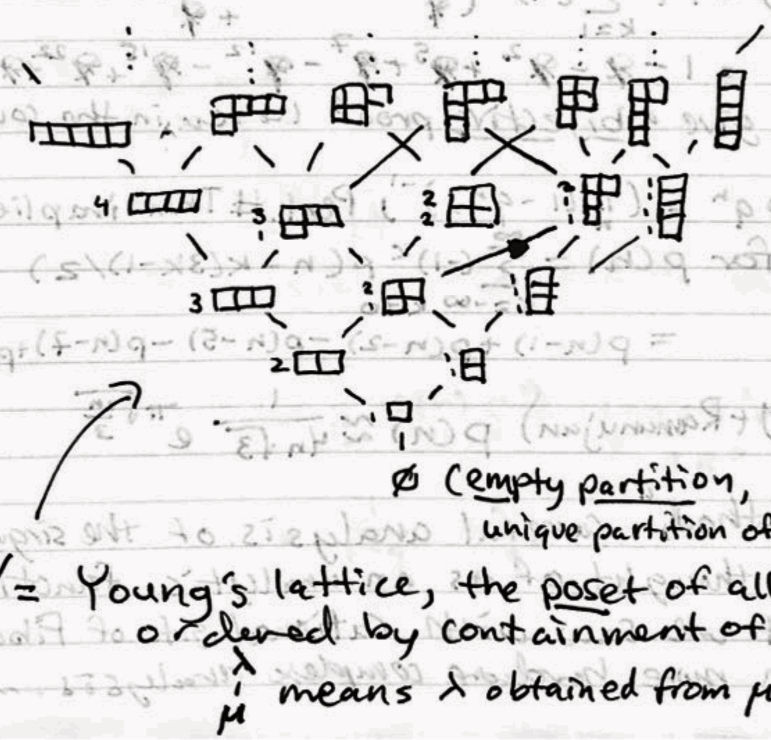
e.g. $\lambda = (5,5,3,1,0,0,\dots) = (5,5,3,1,0) = (5,5,3,1) \vdash 14 = 5+5+3+1$

Its length $\ell(\lambda) := \#\{i : \lambda_i > 0\}$ = # of nonzero parts λ_i.

Its Young diagram is a left + top justified array of boxes, with λ_i boxes in the i^{th} row from the top.

e.g., $\lambda = (5,5,3,1) \leftrightarrow$ 

Let $p(n) :=$ # of partitions $\lambda \vdash n$



p(n)	n
...	...
7	5
5	4
3	3
2	2
1	1
1	0

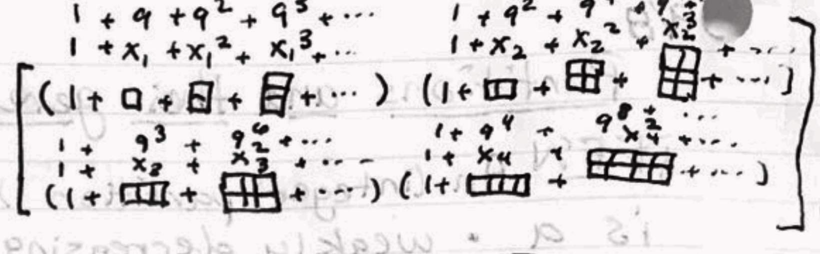
Ø (empty partition, unique partition of 0)

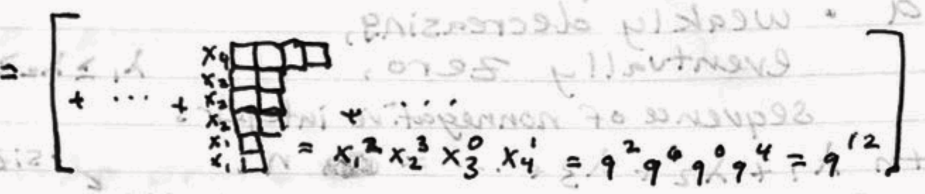
$\mathbb{Y}$ = Young's lattice, the poset of all partitions, ordered by containment of Young diagrams

$\begin{matrix}\lambda \\ | \\ \mu\end{matrix}$ means λ obtained from μ by adding one box

$$\sum_{n\geq 0} p(n)\,q^n = \sum_{\text{all partitions}} q^{|\lambda|} =$$

$\uparrow$ $\mathbb{C}[[q]]$





$$x_i \mapsto q^i \quad = (1+q+q^2+\cdots)(1+q^2+q^4+\cdots)(1+q^3+q^6+\cdots)$$

$$= \frac{1}{1-q}\cdot\frac{1}{1-q^2}\cdot\frac{1}{1-q^3}\cdot\;\cdots \quad \text{a convergent product}$$

$$= \prod_{n=1}^{\infty}\frac{1}{1-q^n} \leftarrow \text{g.f. for all partitions, as a product formula!}$$

Cultural asides:

Thm (Euler's pentagonal number theorem) ← "pentagonal numbers"

$$\prod_{n=1}^{\infty}(1-q^n) = 1+\sum_{k=1}^{\infty}(-1)^k\left(q^{k(3k+1)/2}+q^{k(3k-1)/2}\right)$$

$$= 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-\cdots$$

Pf: We will give a bijective proof later in the course. ▨

Since $\sum p(n)q^n = \left(\prod_n (1-q^n)\right)^{-1}$, Pent. # Thm implies a recurrence for

$$p(n) = \sum_{k=-\infty,\,k\neq 0}^{\infty}(-1)^k\, p(n-k(3k-1)/2)$$

$$= p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-\cdots$$

Thm (Hardy + Ramanujan) $p(n) \approx \frac{1}{4n\sqrt{3}}\, e^{\pi\sqrt{\frac{2n}{3}}}$

Pf: "Circle method," careful analysis of the singularities of $\prod\frac{1}{1-q^n}$ thought of as an analytic function. Like what we saw with rational g.f. of Fibonacci #'s, but much more hardcore complex analysis... ▨

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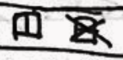
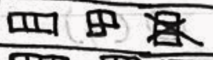
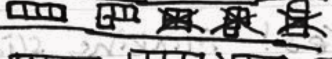
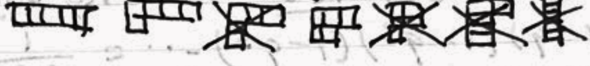
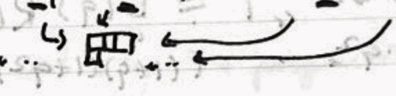
G.f.'s for restricted classes of partitions:

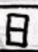
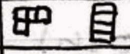
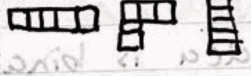
Let $d(n)$:= # of partitions of n into distinct parts.

n	$d(n)$	
0	1	∅
1	1	
2	1	
3	2	
4	2	
5	3	

$$D(q) := \sum_{n\geq 0} d(n)\, q^n = (1+\underline{q})(\underline{1}+q^2)(1+\underline{q}^3)(\underline{1}+q^4)\cdots = \prod_{j=1}^{\infty}(1+q^j)$$



Let $o(n)$:= # partitions of n into odd parts.

n	$o(n)$	
0	1	∅
1	1	
2	1	
3	2	
4	2	
5	3	

Looks like possibly $o(n) = d(n)$, but how to show it?

G.f's!

$$O(q) := \sum_{n\geq 0} o(n)\, q^n = (1+q+q^2+\cdots)(1+q^3+q^6+\cdots)(1+q^5+q^{10}+\cdots)$$

$$= \frac{1}{1-q}\cdot\frac{1}{1-q^3}\cdot\frac{1}{1-q^5}\cdots = \prod_{j\geq 1}^{\infty}\frac{1}{1-q^{2j+1}}$$

$$\overset{?!}{=} \prod_{j\geq 1}^{\infty}(1+q^j)$$

How to show $D(q) = O(q)$?

Well, $D(q) = (1+q)(1+q^2)(1+q^3)\cdots$

$$= \frac{(1-q^2)}{(1-q)} \frac{(1-(q^2)^2)}{(1-q^2)} \frac{(1-(q^3)^2)}{(1-q^3)} \cdots$$

Recall: $(1+x)(1-x) = (1-x^2)$

$$= \frac{(\cancel{1-q^2})(\cancel{1-q^4})(\cancel{1-q^6})(\cancel{1-q^8})\cdots}{(1-q)(\cancel{1-q^2})(1-q^3)(\cancel{1-q^4})\cdots}$$

$$= \frac{1}{(1-q)(1-q^3)(1-q^5)\cdots} = O(q) \, !$$

Was that manipulation ok? Yes! Thinking slightly differently...

Let $R(q) := (1-q)(1-q^3)(1-q^5)\cdots = \frac{1}{O(q)} \in \mathbb{C}[[q]]$

Want to show "$1 \overset{?}{=} D(q)R(q)$ in $\mathbb{C}[[q]]$"

$1+0\cdot q+0\cdot q^2+\cdots$

$$((1+q)(1+q^2)(1+q^3)\cdots)((1-q)(1-q^3)(1-q^5)\cdots)$$

$$= \underbrace{(1+q)(1-q)}_{(1-q^2)}((1+q^2)(1+q^3)\cdots)((1-q^3)(1-q^5)\cdots)$$

$$= (1-q^4)((\underbrace{1+q^3})(1+q^4)\cdots)((\underbrace{1-q^3})(1-q^5)\cdots)$$

$$= \underbrace{(1-q^4)}(1-q^6)(\underbrace{(1+q^4)}_{\cdots}(1+q^5)\cdots)((1-q^5)(1-q^7)\cdots)$$

$$= 1+0\cdot q+0\cdot q^2+0\cdot q^3+\cdots$$

$$= \ldots \text{ et cetera}$$

hinted at by this manipulation

Note: $\exists$ bijective proof that $d(n) = o(n)$ as well

(See Stanley Prop. 1.8.5 for 2 such proofs...) Basic idea is binary expansion:

e.g. $\lambda = (9^5, 5^{12}, 3^2, 1^3) = (9^{2^0+2^2}, 5^{2^2+2^3}, 3^{2^1}, 1^{2^0+2^1})$

$o(n)$

$(9,9,9,9,9,\underbrace{5,5,\ldots,5}_{12},3,3,1,1,1)$

$n=114$

$$\longleftrightarrow \mu = (9\cdot 2^0, 9\cdot 2^2, 5\cdot 2^2, 5\cdot 2^3, 3\cdot 2^1, 1\cdot 2^0, 1\cdot 2^1)$$

$$= (9, 36, 20, 40, 6, 1, 2)$$

(reorder)

$$= (40, 36, 20, 9, 6, 2, 1) \in d(n)$$

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Some useful formal power series

Let's define and study some specific elements of $\mathbb{C}[[x]]$:

DEF'N $e^x := \sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\log(1+x) := \sum_{n\geq 1} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\forall \lambda \in \mathbb{C}$, $(1+x)^\lambda := \sum_{k\geq 0} \binom{\lambda}{k} x^k$ "generalized binomial coefficient"

where $\binom{\lambda}{k} := \frac{\lambda(\lambda-1)(\lambda-2)\cdots(\lambda-(k-1))}{k!} \in \mathbb{C}$

(just like for $n \in \mathbb{N}$, $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-(k-1))}{k!}$.)

These formal power series satisfy the properties you'd expect:

E.g. ① $(1+x)^\lambda (1+x)^\mu = (1+x)^{\lambda+\mu} \in \mathbb{C}[[x]]$ $\forall \lambda,\mu \in \mathbb{C}$

② $e^x e^y = e^{x+y} \in \mathbb{C}[[x,y]]$

③ $e^{\underbrace{\log(1+x)}} = 1 + x$, etc...

defined to be $= 1 + \log(1+x) + \frac{\log(1+x)^2}{2!} + \cdots$

$= 1 + (x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots) + \frac{(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots)^2}{2} + \cdots$

Why does this even converge in $\mathbb{C}[[x]]$?

Prop. If $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, and $b_0 = 0$, then $A(B(x)) := \sum_{n\geq 0} a_n (B(x))^n$ converges in $\mathbb{C}[[x]]$. ⌟

How to justify ①, ②, ③, etc...? Could do a tedious manipulation of coefficients, but instead, since e^x, $\log(1+x)$, $(1+x)^\lambda$ are also analytic functions we are familiar with (whose power series expansions are as above), we can use a trick from complex analysis...

↙ standard fact from complex analysis, true under weaker hypotheses too...

Thm If $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic for $|z| < R$ for some $R > 0$, and f vanishes on $|z| < R$, then $a_0 = a_1 = a_2 = \cdots = 0$.

④ for $n \in \mathbb{N}$, $(1+x)^n = \sum_{n \geq 0} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$,

but also
$$\frac{1}{(1-x)^n} = (1+(-x))^{-n} = \sum_{k\geq 0} \frac{(-n)(-n-1)(-n-2)\cdots(-n-(k-1))}{k!}(-x)^k$$

$$\overset{\text{n terms in product}}{(1+x+\overset{\text{onion}}{\boxed{x^2}}+\cdots)(1+\overset{\text{sesame}}{\boxed{x}}+x^2+\cdots)\cdots(1+x+\overset{\text{asiago}}{\boxed{x^2}}\cdots)} = \sum_{k\geq 0} \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} x^k$$

$**|*||\cdots|*|**$ e.g. $n=18$ flavors of bagels $= \sum_{k\geq 0} \binom{n+k-1}{k} x^k$

Pf that $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$: ('n multichoose k') $\left(\!\binom{n}{k}\!\right) :=$ # k element multisets w/ entries in $\{1,2,\ldots,n\}$

'Stars and bars'
Stars indicate how many of each element is chosen, bars separate bins for each element

e.g. $\{1,1,2,4,4,4,7\}$ is a 7-element multisubset of $\{1,2,\ldots,9\}$

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⑤ $$\underset{(1+(-4x))^{-1}}{\frac{1}{1-4x}} = \sum_{k\geq 0}\binom{-1}{k}(-4x)^k = \sum_{k\geq 0}\underbrace{\binom{1+k-1}{k}}_{=\binom{k}{k}=1}4^k x^k = \sum_{k\geq 0} 4^k x^k \checkmark$$

but also
$$\frac{1}{(1-4x)^2} = \sum_{k\geq 0}\underbrace{\binom{2+k-1}{k}}_{\binom{k+1}{k}=k+1}4^k x^k = \sum_{k\geq 0}(k+1)4^k x^k,$$

$$\frac{1}{(1-4x)^3} = \sum_{k\geq 0} 4^k x^k \cdot \binom{k+2}{2}, \quad \frac{1}{(1-4x)^4} = \cdots \text{ etc.}$$

useful for extracting coefficients of a rational function after performing the Partial fraction expansion ...

⑥ $$\frac{1}{\sqrt{1-4x}} = (1-4x)^{-1/2} = \sum_{k\ge0}\binom{-\frac{1}{2}}{k}(-4)^k x^k$$

$$= \sum_{k\ge0}\left(\frac{(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})\cdots(\frac{-(2k-1)}{2})}{k!}(-4)^k\right)x^k$$

$$= \sum_{k\ge0}\left(\frac{\overset{2^k}{\cancel{4^k}}(1)(3)(5)\cdots(2k-1)}{\cancel{2^k}\;k!}\right)x^k$$

$$= \sum_{k\ge0}\left(\cancel{2^k}\frac{(1)(3)(5)\cdots(2k-1)}{k!}\;\frac{(2)(4)(6)\cdots(2k)}{\cancel{2^k}\;k!}\right)x^k$$

$$= \sum_{k\ge0}\frac{(2k)!}{k!\cdot k!}x^k = \sum_{k\ge0}\binom{2k}{k}x^k$$

this F.P.S. is algebraic but not rational

central binomial coefficients...

interesting...

Another tool from calculus that's useful for $R[[x]]$

DEF'N For $A(x) = \sum_{n\ge0} a_n x^n \in R[[x]]$, we define the formal derivative $A'(x) := \sum_{n\ge1} \underbrace{n\cdot a_n}_{\overset{\shortparallel}{\underbrace{a_n+a_n+\cdots a_n}_{n\text{ times}}}} x^{n-1} \in R[[x]]$

The derivative satisfies the usual rules from calculus:

- $(A(x)+B(x))' = A'(x)+B'(x)$
- $(AB)' = A'\cdot B + B'A$
- $\left(\frac{1}{A}\right)' = \frac{-A'}{A2}$
- $(A(B(x)))' = A'(B(x))\cdot B'(x)$
- etc...

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Quick review of binomial (and multinomial) coefficients

The Binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ has several (easy) interpretations

$= \#$ words with k 1's, $(n-k)$ 2's,

e.g. $\binom{4}{2} = 6 \longrightarrow$

$= \# \{1122, 1212, 1221, 2112, 2121, 2211\}$ i.e., rearrangements of $\underbrace{11\cdots1}_{k}\ \underbrace{22\cdots2}_{n-k}$

$= \#$ lattice paths in $\mathbb{Z}^2$ taking $\overset{E}{\text{east}}$ or $\overset{N}{\text{north}}$ steps, from $(0,0)$ to $(k, n-k)$

e.g.

$\longleftrightarrow$
$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ N & E & E & N & E & E & E & N & E \end{matrix} \leftarrow \binom{9}{6}$

(could be 1's/2's instead of E's/N's)

$= \#$ subsets of $[n] := \{1, 2, \ldots, n\}$ of size k

e.g. $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ N & \underline{E} & \underline{E} & N & \underline{E} & \underline{E} & \underline{E} & N & \underline{E} \end{matrix} \leftrightarrow \{2,3,5,6,7,9\} \subseteq [9]$

(position of E's in word)

Of course, we have

Thm (Binomial Theorem) $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Lemma (Pascal's Identity)

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Pf (bijection):

$A \subseteq [n], \#A = k \mapsto \begin{cases} A & \text{if } n \notin A \subseteq [n-1] \\ A - \{n\} & \text{if } n \in A. \end{cases}$ ☒

$\Rightarrow$ Pascal's triangle:

n=0 1
n=1 1 1
n=2 1 2 1
⋮ 1 3 3 1
1 4 6 4 1
1 5 10 10 5 1 …

Multinomials

How many rearrangements (anagrams) of BANANAS?

i.e., of 3A's, 1B, 2N's, 1S? (equiv. of $\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ A & A & A & B & N & N & S \end{matrix}$)

$\exists$ a transitive action of the symmetric group $\mathfrak{S}_7$ of permutations of $[7]$ on the rearrangements

e.g., perm. $\sigma = \begin{pmatrix} 1&2&3&4&5&6&7 \\ 2&1&4&3&6&7&5 \end{pmatrix}$ sends $\overset{1\,2\,3\,4\,5\,6\,7}{AAABNNS} \xrightarrow{\sigma} \overset{2\,1\,4\,3\,6\,7\,5}{AABANSN}$

The stabilizer of this action is $\underset{\overset{\shortparallel}{\mathfrak{S}_{\{1,2,3\}}}}{\mathfrak{S}_3} \times \underset{\overset{\shortparallel}{\mathfrak{S}_{\{4\}}}}{\mathfrak{S}_1} \times \underset{\overset{\shortparallel}{\mathfrak{S}_{\{5,6\}}}}{\mathfrak{S}_2} \times \underset{\overset{\shortparallel}{\mathfrak{S}_{\{7\}}}}{\mathfrak{S}_1} \subseteq \mathfrak{S}_7$

So by Orbit-Stabilizer Thm, $\#\text{rearrangements} = \text{size of orbit} = \dfrac{\#\mathfrak{S}_7}{\#\mathfrak{S}_3\times\mathfrak{S}_1\times\mathfrak{S}_2\times\mathfrak{S}_1}$ [redacted]

$$= \frac{7!}{3!\,1!\,2!\,1!} = \binom{7}{3,1,2,1}$$

The Multinomial Coefficient $\dbinom{n}{k_1,k_2,\ldots,k_m} = \dfrac{n!}{k_1!k_2!\cdots k_m!}$ for $n = k_1+k_2+\cdots+k_m$

$=$ # words w/ k_1 1's, k_2 2's, …, k_m m's, i.e., rearrang's of $\underbrace{11\cdots1}_{k_1}\underbrace{22\cdots2}_{k_2}\cdots\underbrace{mm\cdots m}_{k_m}$

$=$ # lattice paths in $\mathbb{Z}^m$ taking steps $\underset{\overset{\shortparallel}{(1,0,0,\ldots,0)}}{e_1}$, $\underset{\overset{\shortparallel}{(0,1,0,\ldots,0)}}{e_2}$, …, $\underset{\overset{\shortparallel}{(0,0,\ldots,1)}}{e_m}$ from $0=(0,\ldots,0)$ to $(k_1,k_2,\ldots,k_m)$

(same correspondence between words + walks as w/ binomials)

$=$ # chains $\phi = S_0 \subset S_{k_1} \subset S_{k_1+k_2} \subset \cdots \subset S_{n=k_1+\cdots+k_m} = [n]$ of subsets of $[n]=\{1,2,\ldots,n\}$ for which $\#S_i = i \;\; \forall\, i = 0, k_1, k_1+k_2, \ldots, n$

e.g. for $\binom{7}{3,1,2,1}$ have $2131314 \Leftrightarrow \phi \subset \{2,4,6\} \subset \{1,2,4,6\} \subset \{1,2,3,4,5,6\} \subset [7]$

Note $\dbinom{n}{k} = \dbinom{n}{n,\,n-k}$ in multinomial notation

Also note: $\dbinom{n}{k_1,k_2,\ldots,k_m} = \dbinom{n}{k_1}\cdot\dbinom{n-k_1}{k_2}\cdot\dbinom{n-(k_1+k_2)}{k_3}\cdots\dbinom{n-(k_1+\cdots+k_{m-2})}{k_{m-1}}\dbinom{k_m}{k_m}$

And…

Multinomial Theorem: $$(x_1+x_2+\cdots+x_m)^n = \sum_{\substack{(k_1,k_2,\ldots,k_m) \\ k_1+k_2+\cdots+k_m=n}} \binom{n}{k_1,k_2,\ldots,k_m} x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}$$

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