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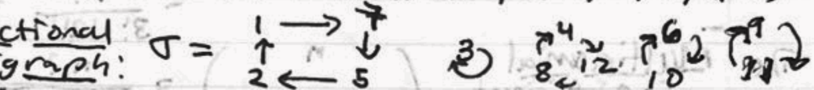
## Permutations and cycles (Stanley § 1.3)

Recall $S_n$ $(= \mathfrak{S}_n)$ = symmetric group on $n$ letters
$= $ permutations of $[n] := \{1, 2, \dots, n\}$
$= \{\text{bijections } \sigma : [n] \to [n]\}$

Notations

- two-line $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 1 & 3 & 12 & 2 & 10 & 5 & 4 & 11 & 6 & 9 & 8 \end{pmatrix}$
- one-line $\sigma = (7, 1, 3, 12, 2, 10, 5, 4, 11, 6, 9, 8)$
- functional digraph: $\sigma =$ 

"digraph" = directed graph

- cycle notation: $\sigma = (1\ 7\ 5\ 2)\ (3)\ (4\ 12\ 8)\ (6\ 10)\ (9\ 11)$
  $= (8\ 4\ 12)\ (10\ 6)\ (5\ 2\ 1\ 7)\ (3)\ (11\ 9)$
  $= \text{etc.} \dots = (3)\ (7\ 5\ 2\ 1)\ (10\ 6)\ (11\ 9)\ (12\ 8\ 4)$

standard form:
- each cycle has its biggest element first
- cycles appear w/ biggest elements increasing left-to-right

cycle type of $\sigma$ = partition of sizes of cycles of $\sigma$

Q: How many $\sigma \in S_n$ of cycle type $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$
$= 1^{c_1} 2^{c_2} 3^{c_3} \dots$

multiplicity notation:
e.g. $\lambda = (5,5,5,3,2,2,2,2,1,1)$
$= 1^2 2^4 3^1 4^0 5^3$,
i.e. $c_1 = 2,\ c_2 = 4,\ c_3 = 1,\ c_4 = 0,\ c_5 = 3$

e.g. $n = 4$

$\lambda = 1^4 =$ (a)(b)(c)(d) $\quad 1$

$2^1 1^2 =$ (ab)(c)(d) $\quad \binom{4}{2} = 6$

$2^2 =$ (ab)(cd) $\quad \binom{4}{2}/2 = 6/2 = 3$

$3^1 1^1 =$ (abc)(d) $\quad 2! \cdot \binom{4}{3} = 2 \cdot 4 = 8$

$4^1 =$ (abcd) $\quad 3! \cdot \binom{4}{4} = 6$

**Prop.:** There are $\dfrac{n!}{1^{c_1} c_1!\, 2^{c_2} c_2!\, 3^{c_3} c_3! \cdots}$ perm.'s in $S_n$ of cycle type $\lambda = 1^{c_1} 2^{c_2} 3^{c_3} \dots$

Pf of prop: Note that $S_n$ acts on the set of perm.'s with cycle type $= \lambda$ transitively, by conjugation:

e.g., $\underbrace{\begin{pmatrix} 1&2&3&4&5&6&7 \\ a&b&c&d&e&f&g \end{pmatrix}}_{\pi} \cdot \underbrace{\left[(1234)(567)\right]}_{\sigma \text{ of type } 4^1 3^1} \cdot \underbrace{\begin{pmatrix} a&b&c&d&e&f&g \\ 1&2&3&4&5&6&7 \end{pmatrix}}_{\pi^{-1}}$

$= (abcd)(efg)$

9/22 So the # of such perm.'s = size of the orbit

$\overset{\text{orbit}}{\underset{\text{stabilizer}}{\hookrightarrow}} = \dfrac{|S_n|}{|Z_{S_n}(\sigma_\lambda)|}$ if $\sigma_\lambda$ is a perm. of type $\lambda$

where $Z_{S_n}(\sigma_\lambda) := \{\pi \in S_n : \pi\sigma_\lambda = \sigma_\lambda\pi\}$ is the centralizer of $\sigma_\lambda$ in $S_n$.
i.e. $\pi\sigma_\lambda\pi^{-1} = \sigma_\lambda$

Which perm.'s centralize $\sigma_\lambda = \underbrace{(a)(b)\cdots}_{c_1 \text{ 1-cycles}} \underbrace{(cd)(ef)\cdots}_{c_2 \text{ 2-cycles}} \cdots$ etc. ?

- Products of powers of each cycle : there are $1^{c_1} 2^{c_2} 3^{c_3} \cdots$ of these
- perm.'s that swap cycles of same size, (preserving cyclic order and biggest element) : there are $c_1!\, c_2!\, c_3! \cdots$ of these
- Products of those : $1^{c_1} c_1!\, 2^{c_2} c_2! \cdots$ many

e.g. $\sigma_\lambda = (1234)(567)(8\,9\,10)$

is centralized by $\pi = \underset{(1234)^3}{(4321)}\ \underbrace{(59)(6\,10)(78)}_{\text{swaps } (567)+(8\,9\,10)}$

Thus $|\text{orbit}| = \dfrac{n!}{\prod_{j\geq 1} j^{c_j} c_j!}$, as claimed. $\blacksquare$

Note: Stanley presents slightly different (but equivalent) proof by considering standard forms of perm.'s $\sigma_\lambda$.

There is an elegant reformulation of above in terms of g.f.'s:

Cor (Touchard) For $\sigma \in S_n$, let $c_k(\sigma) := \#$ of size $k$ cycles of $\sigma$.

$$\sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{\sigma \in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} t_3^{c_3(\sigma)} \cdots\Big) x^n = e^{t_1\frac{x}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots}$$

$$\in \ (\mathbb{C}[t_1, t_2, t_3, \dots])[[x]] \qquad = e^{\sum_{j \geq 1} t_j \frac{x^j}{j}}$$

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Pf: (We'll see a more conceptual proof later...)

$$e^{\frac{t_1 x^1}{1} + \frac{t_2 x^2}{2} + \cdots} = e^{\frac{t_1 x^1}{1}} e^{\frac{t_2 x^2}{2}} \cdots$$

$$= \Big(\sum_{c_1 \geq 0} \frac{\big(\frac{t_1 x^1}{1}\big)^{c_1}}{c_1!}\Big)\Big(\sum_{c_2 \geq 0} \frac{\big(\frac{t_2 x^2}{2}\big)^{c_2}}{c_2!}\Big) \cdots$$

$$= \sum_{(c_1, c_2, \dots)} x^{1 \cdot c_1 + 2 \cdot c_2 + \cdots} \frac{t_1^{c_1} t_2^{c_2} \cdots}{1^{c_1} c_1! \, 2^{c_2} c_2! \cdots}$$

$$= \sum_{n \geq 0} x^n \frac{1}{n!} \sum_{\substack{(c_1, c_2, \dots) \\ \sum_j j \cdot c_j = n}} \frac{n!}{1^{c_1} c_1! \, 2^{c_2} c_2! \cdots} t_1^{c_1} t_2^{c_2} \cdots$$

$\hookrightarrow = \#\{\sigma \in S_n : \forall j,\ \sigma \text{ has } c_j\ j\text{-cycles}\}$ by previous prop.

$$= \sum_{n \geq 0} x^n \frac{1}{n!} \sum_{\sigma \in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots$$

▨

Touchard's theorem has many important consequences, a few of which we now review:

① DEF'N $c(n,k) := \#\{\sigma \in S_n : \sigma \text{ has } k \text{ total cycles}\}$

(Signless) Stirling numbers of 1st kind

I.e., $\sum_{k=1}^{n} c(n,k)\, t^k = \sum_{\sigma \in S_n} t^{\#\text{cycles}(\sigma)}$

Cor (to Touchard) $\sum_{k=1}^{n} c(n,k)\, t^k = t(t+1)(t+2)\cdots(t+(n-1))$

9/27 Pf: Set $t_1 = t_2 = \cdots = t$ in Touchard's thm to get

$$\sum_{n\geq 0} \frac{x^n}{n!} \underbrace{\sum_{\sigma\in S_n} t^{\#\text{cycles}(\sigma)}}_{\sum_{k\geq 1}^{n} c(n,k) t^k} = e^{t\left(\frac{x^1}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right)}$$

$$= e^{t(-\log(1-x))}$$

$$= e^{\log((1-x)^{-t})}$$

$$= (1-x)^{-t}$$

$$= \sum_{n\geq 0} \binom{-t}{n} (-x)^n$$

$$= \sum_{n\geq 0} \underbrace{\binom{t+n-1}{n}}_{=\frac{t(t+1)\cdots(t+(n-1))}{n!}} x^n$$

pf follows by comparing coeff's of $x^n/n!$. ∎

Remark: Prop The map $S_n \to S_n$, $\sigma \mapsto \hat{\sigma}$ = put $\sigma$ in standard cycle form and erase parentheses to view in 1-line not.

is a bijection, w/ #cycles($\sigma$) = # Left-to-right maxima in $\hat{\sigma}$.

Hence, $\sum_{\sigma\in S_n} t^{\#\text{L-to-R max.}(\sigma)} = t(t+1)\cdots(t+(n-1))$

e.g. $n=3$

| $\sigma$ | # L-to-R maxima |
|---|---|
| 1 2 3 | 3 |
| 1 3 2 | 2 |
| 2 1 3 | 2 |
| 2 3 1 | 2 |
| 3 1 2 | 1 |
| 3 2 1 | 1 |

Pf: (by example)

$\sigma \longmapsto \hat{\sigma}$

$(\underline{3})(\underline{7}\,5\,2\,1)(\underline{8}\,4\,6) \qquad \underline{3}\,\underline{7}\,5\,2\,1\,\underline{8}\,4\,6$

is reversible. Just put ( before each L-to-R maxima, and put ) right before the ( and at the end. ∎

② (Cor of Touchard) Can compute $E_K(n) :=$ expected # of k-cycles in a uniformly random $\sigma \in S_n$

$$E_k(n) = \frac{1}{n!}\sum_{\sigma\in S_n} c_k(\sigma) = \frac{1}{n!}\left[\partial/\partial t_k \sum_{\sigma\in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_1=t_2=\cdots=1}$$

$$\text{So } \sum_{n\geq 0} E_k(n)\, x^n = \left[\partial/\partial t_k \sum_{n\geq 0}\frac{x^n}{n!}\sum_{\sigma\in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_1=t_2=\cdots=1}$$

$$= \left[\partial/\partial t_k\, e^{t_1\frac{x^1}{1}+t_2\frac{x^2}{2}+t_3\frac{x^3}{3}+\cdots}\right]_{t_1=t_2=\cdots=1}$$

$$= \left[\frac{x^k}{k} e^{t_1\frac{x^1}{1}+t_2\frac{x^2}{2}+\cdots}\right]_{t_i=1}$$

$$= \frac{x^k}{k} e^{\frac{x^1}{1}+\frac{x^2}{2}+\cdots} = x^k/k\, e^{-\log(1-x)} = \frac{x^k}{(1-x)}$$

$$= \sum_{n\geq k}\frac{1}{k}x^n \Rightarrow E_k(n) = \begin{cases} 1/k & \text{if } n\geq k,\\ 0 & \text{otherwise.}\end{cases}$$

Note: $E_K(n)$ eventually constant in $n$. In fact, one can show #k-cycles of random $\sigma\in S_n$ converges (as $n\to\infty$) to a Poisson random variable w/ expectation $\lambda = 1/k$.

③ (Cor to Touchard) There are special classes of perms defined by restrictions on their cycle sizes, so all have nice g.f.'s.

e.g.: no large cycles

$\sigma\in S_n$ is an involution (i.e. $\sigma^2 = \overset{\text{identity}}{e}$)

$\iff \sigma$ has only 1- and 2-cycles ($\sigma = (ab)(cd)\cdots(x)(y)\cdots$)

$$\text{So } \sum_{n\geq 0}\frac{x^n}{n!}\#\{\text{involutions } \sigma\in S_n\} = \left[e^{t_1\frac{x^1}{1}+t_2\frac{x^2}{2}+\cdots}\right]_{\substack{t_1=t_2=1\\ t_3=t_4=\cdots=0}} = e^{x+\frac{x^2}{2}}$$

and even $\displaystyle\sum_{n\geq 0}\frac{x^n}{n!}\sum_{\substack{\sigma\in S_n\\ \text{involution}}} t^{\#\text{ 1-cycles}(\sigma)} = e^{tx+\frac{x^2}{2}}$, etc.

What about no small cycles?

DEF'N A derangement $\sigma \in S_n$ is a perm. w/ no fixed points, i.e., w/ $c_1(\sigma) = 0$.

Q: (Derangement/ Hat-check problem): $n \geq 100$ people check their hats; attendant gives people's hats back randomly; what is prob. that no one gets their own hat back?

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i.e., what is $\frac{d_n}{n!}$, where $d_n := \# \{\sigma \in S_n : \sigma \text{ is a derangement}\}$?

$$\sum_{n \geq 0} \frac{x^n}{n!} d_n = \left[ e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + \cdots} \right]_{t_1 = 0,\ t_2 = t_3 = \cdots = 1}$$

$$= e^{x^2/2 + x^3/3 + \cdots}$$

$$= e^{-\log(1-x) - \frac{x^1}{1}} = \boxed{\frac{e^{-x}}{1-x}}$$

But $e^{-x}/_{1-x} = (1 + x + x^2 + \cdots)(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots)$

$$= \sum_{n \geq 0} x^n \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

So $\frac{d_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm (-1)^n \frac{1}{n!}$

$\xrightarrow{\text{converges (rapidly) as } n \to \infty} e^{-1} = 1/e \approx 0.368\ldots$

Note: In fact, $d_n =$ closest integer to $\frac{n!}{e}$ $\forall n$.

Also have recurrence for $d_n$:

Prop. $d_n = (n-1) \cdot (d_{n-1} + d_{n-2})$

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# Compositions and their generating functions

<u>DEF'N</u> A <u>composition</u> $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ of $n$, denoted $\alpha \models n$, is a sequence of <u>positive</u> integers $\alpha_i \in \{1,2,3,\dots\}$ w/ $\alpha_1 + \alpha_2 + \dots + \alpha_k = n$. (Unlike a partition, the $\alpha_i$ need not be weakly decreasing.) As w/ partitions, we call $\alpha_i$ the <u>parts</u> of the composition.

<u>e.g.</u> $\alpha = (1,4,2,4)$ is a composition of 11 into 4 parts.

Let $\bar{c}_k(n)$ denote # compositions of $n$ into $k$ parts and $\bar{c}(n)$ denote # compositions of $n$ (into any # of parts).

<u>Prop.</u> $$\sum_{n=0}^{\infty} \bar{c}_k(n) \cdot x^n = \left(\frac{1}{1-x} - 1\right)^k = \left(\frac{x}{1-x}\right)^k$$

<u>Pf:</u> Note $\left(\frac{1}{1-x} - 1\right) = 1 + x + x^2 + x^3 + \cdots - 1 = x + x^2 + x^3 + \cdots$

Now use "picture-writing":

e.g. $k=3$ $\hookrightarrow$ $$(x + \underline{x^2} + x^3 + \cdots)(\underline{x} + x^2 + x^3 + \cdots)(x + x^2 + \underline{x^3} + \cdots)$$

$$\hookrightarrow \cdots + \cdots \quad x^2 \cdot x \cdot x^3 = x^6 + \cdots$$ ∎

<u>Cor.</u> $$\sum_{n=0}^{\infty} \bar{c}(n) \cdot x^n = 1 + \frac{x}{1-2x}.$$

<u>Pf:</u> Note that $\bar{c}(n) = \sum_{k=0}^{\infty} \bar{c}_k(n)$, so

$$\sum_{n=0}^{\infty} \bar{c}(n) \cdot x^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \bar{c}_k(n)\right) \cdot x^n = \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \bar{c}_k(n) \cdot x^n\right)$$

$$= \sum_{k=0}^{\infty} \left(\frac{x}{1-x}\right)^k = \frac{1}{1 - \frac{x}{1-x}}$$

$$= \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x}.$$ ∎

**Cor.** For $n \geq 1$, $\bar{c}(n) = 2^{n-1}$.

**Pf.** $1 + \frac{x}{1-2x} = 1 + \sum_{n\geq 0} 2^n \cdot x^{n+1} = 1 + \sum_{n \geq 1} 2^{n-1} x^n$. ∎

Then extract coeff. of $x^n$ in previous cor.

Since this is such a simple formula, we could ask for a direct proof, not using generating functions. In fact...

**Prop.** The # of compositions of $n$ into $k$ parts is $\bar{c}_k(n) = \binom{n-1}{k-1}$.

**Pf:** Let's say a sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ of <u>nonnegative</u> integers $\alpha_i \in \{0, 1, \ldots\}$ w/ $\sum \alpha_i = n$ is a <u>weak</u> composition of $n$.

**Claim** # of weak compositions of $n$ into $k$ parts $= \left(\!\binom{k}{n}\!\right) = \binom{k+n-1}{n}$ ← recall '<u>multichoose</u>' #

**Pf:** Write a weak composition of $n$, e.g. $\alpha = (2,0,1,3)$ using 'stars and bars'

as $\alpha = **||*|***$.

We saw before that these patterns are counted by $\left(\!\binom{k}{n}\!\right)$.

Finally, $\exists$ a bijection $\{\text{weak comp. of } n \text{ into } k \text{ parts}\} \longleftrightarrow \{\text{(usual) comp. of } n+k \text{ into } k \text{ parts}\}$

$$(\alpha_1, \alpha_2, \ldots, \alpha_k) \mapsto (\alpha_1+1, \alpha_2+1, \ldots, \alpha_k+1).$$

Hence # comp. of $n$ into $k$ parts $= \left(\!\binom{k}{n-k}\!\right) = \binom{k+n-k-1}{n-k} = \binom{n-1}{n-k} = \binom{n-1}{k-1}$. ∎

**Cor.** # comp. of $n$ is $\bar{c}(n) = 2^{n-1}$ for any $n \geq 1$.

**Pf:** $\bar{c}(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$. ✓ ∎