10/1

Some theory of ordinary generating functions (Ardila §2.2)

Roughly speaking, if $\mathcal{A}$ is a class of combinatorial structures, w/ $a_n = \#$ (weighted?) $\mathcal{A}$-structures of "size" n ($\in$ ring R), then we can form the ordinary generating function $A(x) = \sum_{n \geq 0} a_n x^n \in R[[x]]$

Prop.
- If $\mathcal{C}$ structures of size n are a choice of either an $\mathcal{A}$-structure or $\mathcal{B}$-structure of size n ("$\mathcal{C} = \mathcal{A} + \mathcal{B}$") (i.e., $c_n = a_n + b_n$) then $C(x) = A(x) + B(x)$.
- If $\mathcal{C}$ structures of size n are a choice of
  - an $\mathcal{A}$-structure of size i
  - a $\mathcal{B}$-structure of size j ("$\mathcal{C} = \mathcal{A} \times \mathcal{B}$")

  for some $i + j = n$ (i.e., $c_n = \sum_{\substack{i+j=n \\ i,j \geq 0}} a_i b_j$)

  then $C(x) = A(x) \cdot B(x)$.
- If $\mathcal{C}$-structures of size n are a choice of $\mathcal{B}$-structures of sizes $i_1, i_2, \ldots, i_k$ for some $i_1 + i_2 + \cdots + i_k = n$, $i_j \geq 0$, for some $k \geq 0$

  (i.e., $c_n = \sum_{\substack{(i_1, i_2, \ldots, i_k) \\ \Sigma i_j = n \\ i_j \geq 0}} b_{i_1} b_{i_2} \cdots b_{i_k}$) ("$\mathcal{C} = Seq(\mathcal{B})$")

  "sequences of $\mathcal{B}$-structures"

  then $C(x) = \dfrac{1}{1 - B(x)}$.

Pf of the proposition is straightforward (just uses definition of addition + multiplication of FPS's).

Now let's see some examples of how to apply this...

EXAMPLES (see also Ardila § 2.2.2)

① (Partitions w/ bounded part sizes)

Let $P_{\leq k}(n) := \#\{\text{partitions } \lambda = (\lambda_1, \lambda_2, \ldots) \vdash n : \lambda_1 \leq k\}$,

i.e., partitions of n into parts of size at most k.

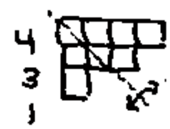
Then $$P_{\leq k}(q) = \sum_{n \geq 0} P_{\leq k}(n)\, q^n = \sum_{\lambda : \lambda_1 \leq k} q^{|\lambda|}$$

$$= \underbrace{\frac{1}{1-q}}_{\substack{\text{o.g.f. for } \lambda \\ \text{w/ only parts} \\ \text{of size} = 1}} \cdot \underbrace{\frac{1}{1-q^2}}_{\substack{\text{o.g.f for } \lambda \\ \text{w/ only parts} \\ \text{of size} = 2}} \cdot \cdots \cdot \underbrace{\frac{1}{1-q^k}}_{\substack{\text{o.g.f for} \\ \lambda \text{ w/ only} \\ \text{parts size} = k}} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$$

↑ we saw the $k \to \infty$ limit of this before

i.e., $\mathcal{C} = \{\lambda : \lambda_1 \leq k\} = \mathrm{Seq}(\text{Ones}) \times \mathrm{Seq}(\text{Twos}) \times \cdots \times \mathrm{Seq}(k\text{'s})$

Remark The conjugate (or transpose) of a partition λ, denoted λ^t, is the partition whose Young diagram is the reflection of the Young diagram of λ across 'main diagonal':

$\lambda =$  $\longleftrightarrow$ $\lambda^t =$ 

Conjugation is a bijection (in fact, involution!) on partitions, and $\lambda_1 = \ell(\lambda^t)$. ← recall length $\ell = \#$ of parts.

Hence also $P_{\leq k}(n) = \#\{\text{partitions } \lambda \vdash n : \ell(\lambda) \leq k\}$.

And so we also have $$\sum_{\lambda : \ell(\lambda) \leq k} q^{|\lambda|} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$$

Similarly, $$\sum_{\substack{\lambda : \\ \lambda_1 \leq k}} q^{|\lambda|} t^{\ell(\lambda)} = \frac{1}{(1-tq)(1-tq^2)\cdots(1-tq^k)} = \sum_{\substack{\lambda : \\ \ell(\lambda) \leq k}} q^{|\lambda|} t^{\lambda_1}.$$

② (Compositions)

Recall $\bar{c}(n) = \#$ compositions $\alpha \models n$.

We have $\sum_{n\geq 0} \bar{c}(n) x^n = \dfrac{1}{1-\underbrace{(x+x^2+x^3+\cdots)}_{\substack{\text{o.g.f. for}\\ \text{compositions of } n\\ \text{w/ one part, since}\\ \exists \text{ unique such comp.}}}} = \cdots = 1 + \underbrace{\dfrac{x}{1-2x}}_{\text{saw this previously}}$.

i.e., "$\mathcal{C} = Seq\left(\begin{smallmatrix}\text{one-part}\\ \text{compositions}\end{smallmatrix}\right)$"

③ (Partitions/compositions w/ restricted parts)

Generalizing previous two examples, let $S \subseteq \{1,2,3,\ldots\}$ be any subset of positive integers.

First consider $\mathcal{C} = \{\text{partitions } \lambda : \text{all parts } \lambda_i \in S\}$.

Then $\mathcal{C} = \prod_{j\in S} Seq(j\text{'s})$, so

$$C(x) = \sum_{n\geq 0} \#\{\lambda \vdash n : \text{all } \lambda_i \in S\} x^n = \prod_{j\in S} \frac{1}{1-x^j}.$$

Next consider $\mathcal{C} = \{\text{compositions } \alpha : \text{all parts } \alpha_i \in S\}$.

Then $\mathcal{C} = Seq(\text{one-part compositions } \alpha_1 \text{ w/ } \alpha_1 \in S)$, so

$$C(x) = \sum_{n\geq 0} \#\{\alpha \models n : \text{all } \alpha_i \in S\} x^n = \frac{1}{1-\sum_{j\in S} x^j}.$$

e.g. if $S = \{1,2\}$, then

$$\sum_{n\geq 0} \#\{\text{comp. of } n \text{ into 1's and 2's}\} x^n = \frac{1}{1-(x+x^2)},$$

which we saw on 1st week of class, with the Fibonacci numbers!

10/4

④ (Stirling #'s of the 2^{nd} Kind)

DEF'N A set partition of $[n]$ is a set $\pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ of subsets $\pi_i \subseteq [n]$ s.t.
- (nonempty) $\pi_i \neq \emptyset \ \forall i$
- (disjoint) $\pi_i \cap \pi_j = \emptyset \ \forall i \neq j$
- (covering) $\bigcup \pi_i = [n]$

The π_i are called the blocks of the set partition π.

DEF'N $S(n,k) :=$ # of set partitions of $[n]$ into exactly k blocks
← "Stirling #'s of the 2^{nd} kind"

e.g. 1|2|3 | 12|3, 13|2, 23|1 | 123
$S(3,3)=1$ | $S(3,2)=3$ | $S(3,1)=1$

Table of $S(n,k)$:

n \ k	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	1	3	1	0
4	1	7	6	1

(Pascal-like) Recurrence for $S(n,k)$:

$$S(n,k) = \underbrace{S(n-1,k-1)}_{\text{n is in a singleton block}} + \underbrace{k \cdot S(n-1,k)}_{\text{n goes into one of the k other blocks}} \quad \text{for } k>1$$

w/ initial conditions $S(n,1) = 1 \ \forall n$
$S(0,0) = 1$, $S(n,k) = 0$ if $k > n$

Let's study the o.g.f. $F_k(x) = \sum_{n\geq 0} S(n,k)\, x^n$ in 2 ways:

ⓐ Solve recurrence: for $k \geq 2$,

$$\sum_{n\geq 0} S(n,k)\, x^n = \sum_{n\geq 0} S(n-1,k-1)\, x^n + \sum_{n\geq 0} k \cdot S(n-1,k)\, x^n$$

$$F_k(x) = x\, F_{k-1}(x) + k\, x\, F_k(x)$$

$$(1-kx) F_k(x) = x\, F_{k-1}(x)$$

$$\boxed{F_k(x) = \frac{x}{1-kx} F_{k-1}(x)}$$

(and for $k=1$, $F_1 = \sum_{n\geq 0} S(n,1)\, x^n = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$)

$$\Rightarrow F_k(x) = \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x} \cdots \frac{x}{1-2x} \cdot \frac{x}{1-x} = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

(b) Let $\mathcal{A}_m :=$ the structure of strings of letters from $[m]$ that start w/ an m, whose size is length

(e.g. $m=3$ $\underbrace{31312}_{\text{size } 5}$ or $\underbrace{3311}_{\text{size } 4}$)

Prop. $\left\{ \begin{array}{l} \text{Set partitions} \\ \pi \text{ of } [n] \text{ w/ } k \text{ blocks} \end{array} \right\} \xleftrightarrow{f} \left\{ \begin{array}{l} \text{(total) size} = n \text{ structures} \\ \text{in } \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k \end{array} \right\}$

$\pi \longmapsto$ the restricted growth function $f : [n] \to [k]$ associated to π

e.g. $n=16$, $k=4$

$$\overset{①}{1,2,4,5,8,12} \,\Big|\, \overset{②}{3,6,9,10} \,\Big|\, \overset{③}{7,11,16} \,\Big|\, \overset{④}{13,15} \mapsto$$

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(i)$	1	1	2	1	1	2	3	1	2	2	3	1	4	2	4	3

$\in \mathcal{A}_1$, $\in \mathcal{A}_2$, $\in \mathcal{A}_3$, $\in \mathcal{A}_4$

number the blocks of π ①, ②, ..., ⓚ according to increasing smallest elements

$f(i) :=$ block # containing i

Pf: Exercise for you ...

Cor $$F_k(x) = \underbrace{\frac{x}{1-x}}_{\text{o.g.f. for } \mathcal{A}_1 \,=\, x+x^2+x^3+\cdots} \cdot \underbrace{\frac{x}{1-2x}}_{\text{o.g.f. for } \mathcal{A}_2 \,=\, x+2x^2+4x^3+\cdots} \cdots\cdots \underbrace{\frac{x}{1-kx}}_{\text{o.g.f. for } \mathcal{A}_k \,=\, x+kx^2+k^2x^3+\cdots} = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$$

10/6 The two kinds of Stirling #'s

How are $\underbrace{S(n,k)}_{\text{Stirling \#'s of 2nd kind}}$ and $\underbrace{c(n,k)}_{\text{(signless) Stirling \#'s of 1st kind}}$ related?

$c(n,k) = \#\{\sigma \in S_n : \sigma \text{ has } k \text{ cycles}\}$

Note: The $c(n,k)$ satisfy a similar (Pascal-like) recurrence:

$$c(n,k) = \underbrace{c(n-1,k-1)}_{n \text{ is in a 1-cycle}} + \underbrace{(n-1)\,c(n-1,k)}_{n \text{ maps to some } i \in [n-1]}$$

$c(n,k)$

$n \backslash k$	1	2	3	4
1	1	0	0	0
2	1	1	0	0
3	2	3	1	0
4	6	11	6	1

But the real connection between Stirling #'s is ...

Prop. (i) $x^n = \sum_{k=1}^{n} S(n,k)\,(x)_k$ where $(x)_k := x(x-1)(x-2)\cdots(x-(k-1))$

while (ii) $(x)_n = \sum_{k=1}^{n} s(n,k)\, x^k$

$s(n,k) = (-1)^{n-k} c(n,k)$ (= (signed) Stirling #'s of 1st kind)

Hence (iii) the infinite lower-triangular matrices

$(S(n,k))_{\substack{k=1,2,\ldots \\ n=1,2,\ldots}}$ and $(s(n,k))_{\substack{k=1,2,\ldots \\ n=1,2,\ldots}}$

are inverses of one another,

i.e., (iv) $\sum_{k=1}^{n} S(n,k)\, s(k,m) = \delta_{n,m} = \sum_{k=1}^{n} s(n,k)\, S(k,m)$.

Kroenecker delta $= \begin{cases} 1 & \text{if } n=m \\ 0 & \text{otherwise.} \end{cases}$

e.g.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}}_{S(n,k)} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{bmatrix}}_{s(n,k)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

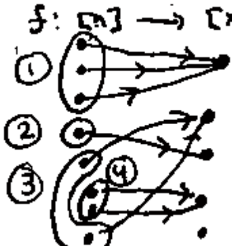
Pf: For (i), it is enough to prove when x is a nonnegative integer (since it is an identity of polynomials ...)

For $x = 1, 2, 3, \ldots$

$$x^n = \#\left\{ \text{functions } [n] \xrightarrow{f} [x] \right\} = \sum_{\substack{\text{set partitions} \\ \pi \text{ of } [n]}} \#\left\{ f : [n] \to [x] \text{ for which set of preimages } \{f^{-1}(i)\}_{\substack{i \in [x] \\ \neq \emptyset}} = \pi \right\}$$

(non-empty)

$f: [n] \to [x]$



$\{①, ②, ③, ④\} = \pi$ associated to f

$$= \sum_{k=1}^{n} S(n,k) \overbrace{x(x-1)(x-2)\cdots(x-(k-1))}^{(x)_k}$$

choice of which $i \in [x]$ are images of the (non-empty) preimages determined by π

For (ii), recall that $x(x+1)(x+2)\cdots(x+(n-1)) = \sum_{k=1}^{n} c(n,k)\, x^k$

$x \mapsto (-x)$ and mult. by $(-1)^n$ $\rightsquigarrow$ $x(x-1)(x-2)\cdots(x-(n-1)) = \sum_{k=1}^{n} s(n,k)\, x^k$.

Then (iii) follows, because (i) and (ii) say that $S(n,k)$ and $s(n,k)$ are coeff's in transition between bases $\{x^n\}$ and $\{(x)_n\}$ of $\mathbb{C}[x]$. ∎

& (iv)

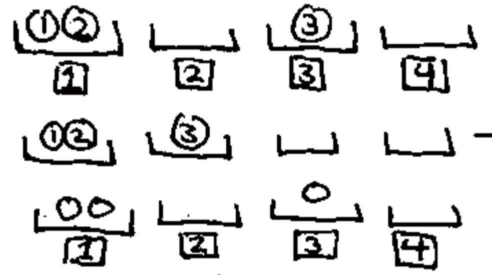
10/8

The twelvefold way (Stanley §1.9)

By now we've seen many examples of counting how to put n balls into k boxes. The 12-fold way is a systematic approach to those kinds of problems, where:

- the balls can be distinguishable or indistinguishable,
- the boxes can also be dist. or indist.,
- the assignment of balls to boxes can be:
  - i) arbitrary,
  - ii) injective, i.e., at most one ball per box,
  - iii) surjective, i.e., at least one ball per box.

e.g. $n=3$, $k=4$



→ dist. balls / dist. boxes

→ dist. balls / indist. boxes

→ indist. balls / dist. boxes

Altogether, we get $12 = 2\times 2\times 3$ possibilities.
(Formally, we can view assignments as functions $f: \underset{\text{balls}}{[n]} \to \underset{\text{boxes}}{[k]}$ and making balls/boxes indist. corresponds to "modding out" by S_n/S_k action on domain/codomain.)

12-fold way	Any f	Injective f	Surjective f
Dist. Balls Dist. Boxes	1. k^n	2. $(k)_n = k(k-1)\cdots(k-(n-1))$	3. $k!\cdot S(n,k)$
Indist. Balls Dist. Boxes	4. $\left(\!\binom{k}{n}\!\right) = \binom{k+n-1}{n}$	5. $\binom{k}{n}$	6. $\left(\!\binom{k}{n-k}\!\right) = \binom{n-1}{k-1}$
Dist. Balls Indist. Boxes	7. $\sum_{j=0}^{k} S(n,j)$	8. $\begin{cases} 1 & \text{if } n\le k \\ 0 & \text{if } n>k \end{cases}$	9. $S(n,k)$
Indist. Balls Indist. Boxes	10. $\sum_{j=0}^{k} P_j(n) = P_k(n+k)$	11. $\begin{cases} 1 & \text{if } n\le k \\ 0 & \text{if } n>k \end{cases}$	12. $P_k(n)$

<u>Explanation of all these formulas:</u>

1. (Dist balls + boxes, Any f): This very basic case was on the HW.
For each ball, we can choose one of k boxes $\to k^n$

2. (Dist balls + boxes, injective): Similar to previous case, but now choose boxes for balls ①, ②, …, ⓝ in order: first ball has k boxes, 2^{nd} has $(k-1)$ because needs to be different from 1^{st}, etc. $\to k(k-1)\cdots(k-(n-1))$

3. (Dist balls + boxes, surjective): This determines an <u>ordered</u> <u>set partition</u> $(\pi_1, \pi_2, \ldots, \pi_k)$ of $[n]$: $\to$ $\pi_1 = \{1,3\}$, $\pi_2 = \{5\}$, $\pi_3 = \{2,4\}$
of <u>ordered</u> s.p. of $[n]$ into k blocks $= k! \cdot$ # (unordered) s.p. into k blocks
$= k! \cdot S(n,k)$ why?

4. (Indist. balls / dist. boxes, any): This determines a <u>weak</u> <u>composition</u> of n into k parts: $\to ** \,|\,|\, * \,|\, ***$.
We've seen using stars + bars why $\binom{k+n-1}{n}$ is right answer.

5. (I. balls / d. boxes, inj.): Choose boxes that get a ball $\to \binom{n}{k}$

6. (I. balls / d. boxes, sur.): This determines a <u>composition</u> of n into k parts: $\to$ we saw before ans. $= \binom{n-1}{k-1}$.

7. (D. balls / I. boxes, any): This determines a set partition of $[n]$ into <u>at</u> <u>most</u> k blocks: $\to S(n,0) + S(n,1) + \cdots + S(n,k)$.
(because some boxes may be empty!)

8. (D. balls, I. boxes, inj.): Only possibility here looks like: …, which exists only if $n \le k$.

10. (I. balls + boxes, any): This determines a <u>partition</u> of n into <u>at</u> <u>most</u> k parts: (because boxes may be empty!)
So if $P_k(n) =$ # partitions of n into exactly k parts, answer is $P_0(n) + P_1(n) + P_2(n) + \cdots + P_k(n) = P_k(n+k)$, as you saw on HW.

11. (I. balls + boxes, any): Only possibility: …, which exists only if $n \le k$

12. (I. balls + boxes, inj.): This is partition of n into k parts $\to P_k(n)$.

9. (D. balls / I. boxes, inj.): This is a set partition of $[n]$ into k blocks $\to S(n,k)$

10/11

Catalan numbers (See Stanley's other book on this!)

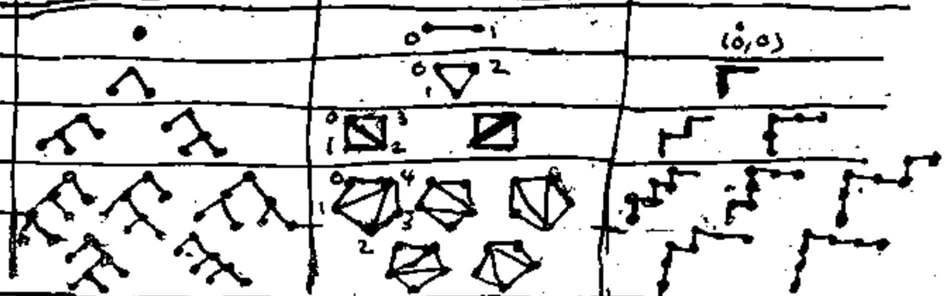
$C_n :=$ n^{th} Catalan number $= \#\{$plane binary trees w/ $n+1$ leaves (or n internal vertices, each w/ a L+R child$\}$

$= \#\{$triangulations of $(n+2)$-gon$\}$

$= \#\{$lattice paths taking N, E steps $(0,0)\to(n,n)$, staying (weakly) above diagonal $y=x\}$

e.g.

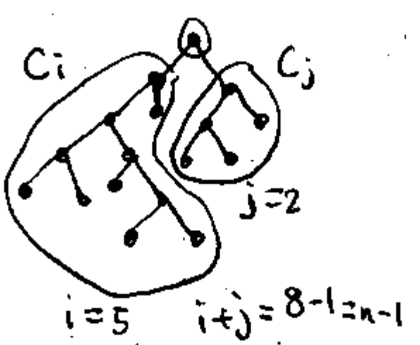
n	C_n	plane binary trees	triangulations	lattice paths
0	1			
1	1			
2	2			
3	5			

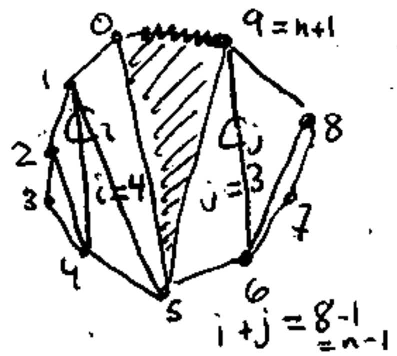


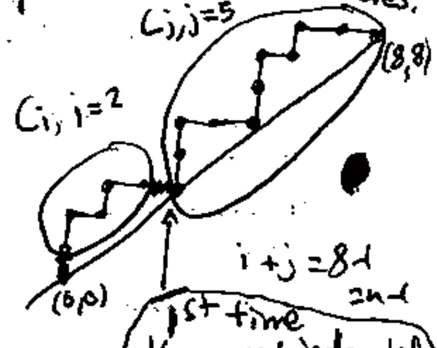
Prop. (Fundamental recurrence)

$$C_n = \sum_{i+j=n-1} C_i \cdot C_j \quad \text{for } n \geq 1.$$

Pf: Each structure 'decomposes' as a product of two smaller ones:







Cor Setting $C(x) := \sum_{n\geq 0} C_n x^n$, we have

$$x\,C(x)^2 - C(x) + 1 = 0 \overset{\text{(quadratic thm)}}{\Rightarrow} C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Pf: Fund. recurrence $C_n = \sum_{i+j=n-1} C_i \cdot C_j$ translates to

$$C(x) = 1 + \sum_{n\geq 1} C_n x^n = 1 + \sum_{n\geq 1} \Big(\sum_{i+j=n-1} C_i \cdot C_j\Big) x^n$$
$$= 1 + x \cdot \sum_{n\geq 0} \Big(\sum_{i+j=n} C_i \cdot C_j\Big) x^n$$
$$= 1 + x \cdot C(x)^2. \qquad \blacksquare$$

Thm $C_n = \frac{1}{n+1}\binom{2n}{n} \left(= \frac{(2n)!}{n!\,(n+1)!} = \frac{1}{2n+1}\binom{2n+1}{n} \right)$

Pf: Recall that $\frac{1}{\sqrt{1-4x}} = (1-4x)^{-1/2} = \sum_{n\geq 0} \binom{2n}{n} x^n$.

Integrate to get $-\frac{1}{2}(1-4x)^{1/2} = K + \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{n+1}$

Check $x=0 \Rightarrow K = -\frac{1}{2} \Rightarrow$

$$\frac{1}{2} - \frac{1}{2}(1-4x)^{1/2} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{n+1}$$

$$\frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n.$$

Since $C(x) = \frac{1\pm\sqrt{1-4x}}{2x}$, and $\frac{1-\sqrt{1-4x}}{2x}$ has positive coeff's,

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n \Rightarrow C_n = \frac{1}{n+1}\binom{2n}{n}. \qquad \blacksquare$$

Can check $1 = \frac{1}{1}\binom{0}{0}$, $1 = \frac{1}{2}\binom{2}{1}$, $2 = \frac{1}{3}\binom{4}{2}$, $5 = \frac{1}{4}\binom{6}{3}$, ...
and next term is $14 = \frac{1}{5}\binom{8}{4} = \frac{1}{5}\cdot 70$, so there are 14 triangulations of a hexagon (and 42 triang's of 7-gon)!