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Exponential generating functions (Ardila § 2.3)

$\mathcal{A}$ = structure one can place on labelled objects like $[n]$

a_n = # of such structures one can place on $[n]$

$\rightsquigarrow A(x) := \sum_{n\geq 0} a_n \frac{x^n}{n!}$ =: the exponential gen. fun. for $\mathcal{A}$.

Prop.
- If $\mathcal{C}$ structures are a choice of $\mathcal{A}$- or $\mathcal{B}$-structures then $C(x) = A(x) + B(x)$, ("$\mathcal{C} = \mathcal{A} + \mathcal{B}$")
- If $\mathcal{C}$ structures on $[n]$ are a choice of partition $[n] = S_1 \sqcup S_2$, with an $\mathcal{A}$-structure on S_1, $\mathcal{B}$-structure on S_2, ("$\mathcal{C} = \mathcal{A} * \mathcal{B}$") so that $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$, then $C(x) = A(x) \cdot B(x)$.
- If $\mathcal{C}$-structures are a choice of (unordered) set partition π of $[n]$, and then an $\mathcal{A}$-structure on each block of π, then $\boxed{C(x) = e^{A(x)}}$. ("$\mathcal{C} = \mathrm{Set}(\mathcal{A})$")

technical pt.: need $a_0 = 0$ here

"The exponential formula"

<u>Pf</u>:
- $\mathcal{C} = \mathcal{A} + \mathcal{B}$ is obvious.
- For $\mathcal{C} = \mathcal{A} * \mathcal{B}$, note $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \Leftrightarrow \frac{c_n}{n!} = \sum_{i+j=n} \frac{a_i}{i!} \frac{b_j}{j!}$
$\Leftrightarrow C(x) = A(x) B(x)$. ✓

- For $\mathcal{C} = \mathrm{Set}(\mathcal{A})$, note $\mathcal{C} = \bigsqcup_{k=0}^{\infty} \mathcal{A}^{(k)}$, where $\mathcal{A}^{(k)} = \{$pick a set partition of $[n]$ into exactly k blocks, put $\mathcal{A}$-structure on each block$\}$

So $C(x) = \sum_{k=0}^{\infty} A^{(k)}(x)$.

But $k! A^{(k)}(x) = A(x)^k$ = e.g.f for $\underbrace{\mathcal{A} * \mathcal{A} * \cdots * \mathcal{A}}_{k \text{ times}}$
= $\{$pick a set partition $[n] = B_1 \sqcup \cdots \sqcup B_k$ into k ordered blocks, and put $\mathcal{A}$-structure on each block$\}$

$\boxed{\text{note: } a_0 = 0 \Rightarrow \text{all } B_i \neq \emptyset}$

Hence, $A^{(k)}(x) = \frac{A(x)^k}{k!}$,

and so $C(x) = \sum_{k=0}^{\infty} \frac{A(x)^k}{k!} = e^{A(x)}$. ✓ ∎

Examples of e.g.f.'s:

hat-check problem

① Recall $d_n = \#\{\text{derangements in } S_n\}$, $D(x) := \sum_{n\geq 0} \frac{d_n}{n!} x^n$

$$\{\text{all permutations}\} = \left\{\begin{matrix}\text{fixed point only perms,} \\ \text{i.e., identity perms}\end{matrix}\right\} \times \left\{\begin{matrix}\text{derangements,} \\ \text{(fixed-pt-free perms)}\end{matrix}\right\}$$

So $\sum_{n\geq 0} \cancel{n!} \frac{x^n}{\cancel{n!}} = \left(\sum_{n\geq 0} 1 \cdot \frac{x^n}{n!}\right) \cdot D(x)$

$$\frac{1}{1-x} = e^x \cdot D(x), \text{ i.e., } D(x) = \frac{e^{-x}}{1-x}, \text{ as we saw.} \checkmark$$

② $\left\{\begin{matrix}\text{involutions} \\ \sigma^2 = 1\end{matrix}\right\} = \mathrm{Set}\left(\left\{\begin{matrix}\text{involutions w/} \\ \text{exactly one cycle}\end{matrix}\right\}\right)$

Hence $\sum_{n\geq 0} \#\left\{\begin{matrix}\sigma \in S_n: \\ \sigma^2 = 1\end{matrix}\right\} \frac{x^n}{n!} = e^{\sum_{n\geq 0} \#\{\sigma \in S_n: \sigma^2 = 1, \#\text{cycles}(\sigma) = 1\} \frac{x^n}{n!}}$

$$= e^{0 \cdot \frac{x^0}{0!} + 1 \cdot \frac{x^1}{1!} + 1 \cdot \frac{x^2}{2!} + 0 \cdot \frac{x^3}{3!} + 0 \cdot \frac{x^4}{4!} + \cdots}$$

$$= e^{x + \frac{x^2}{2}}, \text{ as we saw before.} \checkmark$$

③ More generally, Touchard's Thm follows from exp. formula:

$$\{\text{permutations}\} = \mathrm{Set}(\{\text{permutations w/ exactly one cycle}\})$$

and if we weight σ by $t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots$, wt is multiplicative with respect to this decomposition.

So $\sum_{n\geq 0} \frac{x^n}{n!} \left(\sum_{\sigma \in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots\right) = e^{\sum_{n\geq 0} \frac{x^n}{n!} \left(\sum_{\substack{\sigma \in S_n: \\ \sigma \text{ has only} \\ \text{one cycle}}} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots\right)}$

$$= e^{\sum_{n\geq 1} \frac{x^n}{n!} \cdot t_n \cdot (n-1)!}$$

there are $(n-1)!$ n-cycles in S_n $(1\, \underbrace{a_1\, a_2 \cdots a_{n-1}}_{\text{arbitrary seq.}})$

$$= e^{\sum_{n\geq 1} t_n \frac{x^n}{n}}$$

$$= e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots}, \text{ as we saw.} \checkmark$$

In addition to permutations, e.g.f.'s are also useful for set partitions and graphs / trees ...

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④ <u>Bell numbers</u> $B_n := \#\{\text{set partitions of } [n]\}$
+ <u>Bell polynomials</u> $B_n(y) := \sum_{\substack{\text{set partition} \\ \pi \text{ of } [n]}} y^{\#\text{blocks}(\pi)} = \sum_{k=0}^{n} S(n,k)\, y^k$.

Since $\{\text{set partitions}\} = \text{Set}(\{\text{single (non-empty) block partitions}\})$

$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{1\cdot\frac{x}{1!} + 1\cdot\frac{x^2}{2!} + 1\cdot\frac{x^3}{3!} + \cdots} = e^{(e^x - 1)}$$

and

$$\sum_{n\geq 0} B_n(y) \frac{x^n}{n!} = e^{y\cdot\frac{x}{1!} + y\cdot\frac{x^2}{2!} + y\cdot\frac{x^3}{3!} + \cdots} = e^{y(e^x - 1)}$$

<u>Cor</u> (extract coeff. of $[y^k]$) $\Rightarrow \sum_{n\geq 0} S(n,k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$

⑤ Let's count <u>connected</u>, <u>simple</u> <u>graphs</u> $G = (V^{\shortparallel [n]}, E^{\subseteq \binom{[n]}{2}})$
weighted by $y^{\#E}$ (number of edges)

<u>e.g.</u> <u>n=3</u>

$\Rightarrow 3y^2 + y^3$

Can we understand $\text{Conn}(x,y) := \sum_{n\geq 1} \frac{x^n}{n!} \sum_{\substack{\text{connected} \\ \text{graphs } G \text{ on } [n]}} y^{\#E}$?

<u>Note</u>: $\left\{\begin{matrix}\text{all simple} \\ \text{graphs}\end{matrix}\right\} = \text{Set}(\{\text{connected simple graphs}\})$

So $\text{All}(x,y) = e^{\text{Conn}(x,y)}$

$$\Rightarrow \text{Conn}(x,y) = \log(\text{All}(x,y)) = \log\Big(\sum_{n\geq 0} \frac{x^n}{n!} \sum_{\substack{\text{simple} \\ \text{graphs on } [n]}} y^{\#E}\Big)$$

$$= \log\Big(1 + \sum_{n\geq 1} \frac{x^n (1+y)^{\binom{n}{2}}}{n!}\Big) \quad \leftarrow \boxed{\text{include, or not, each edge}}$$

computer

$$\overset{\downarrow}{=} x + \frac{x^2}{2!}\cdot 1\cdot y + \frac{x^3}{3!}(3y^2 + y^3) + \frac{x^4}{4!}(16y^3 + 15y^4 + 6y^5 + y^6) + \cdots$$

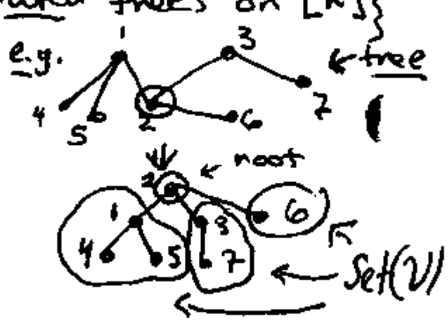
⑥ Let's try to understand $t_n := \#\{\underline{\text{trees}} \text{ on } [n]\}$

n	trees	t_n
1	$\cdot 1$	1
2	1—2	1
3	∠ ⅃ ∧	3
4	...	16
5	...	125

If we set $V_n := \#\{\underline{\text{vertex-rooted}} \text{ trees on } [n]\}$
then $V_n = n \cdot t_n$
and $\mathcal{V} = \{\text{root}\} * \text{Set}(\mathcal{V})$
so that $\boxed{V(x) = x e^{V(x)}}$

$\underbrace{V(x)}_{\sum_{n\geq 0} V_n \frac{x^n}{n!}}$

<u>e.g.</u>



Is $V(x) = xe^{V(x)}$ useful? Yes! Can rephrase as
$V(x)\,e^{-V(x)} = x$, i.e., $V(x)$ is compositional inverse of xe^{-x}.

Prop. If $A(x) = a_1x + a_2x^2 + \cdots \in R[[x]]$ has zero constant term ($a_0 = 0$), so that $B(A(x))$ is well-defined, then A has a compositional inverse $B = A^{\langle -1\rangle}$, satisfying $B(A(x)) = A(B(x)) = x \iff a_1 \in R^\times$ is a unit.

But why does knowing $V(x) = A^{\langle -1\rangle}(x)$ for $A(x) = xe^{x}$ help?

Lagrange inversion thm:
If $B(x) = A^{\langle -1\rangle}(x)$, that is $B(A(x)) = x$ for some $A(x), B(x) \in \mathbb{C}[[x]]$
then $[x^n]\,B(x) = \frac{1}{n}[x^{-1}]\left(\frac{1}{A(x)^n}\right) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{A(x)}\right)^n$.

Let's see how Lagrange inversions solves tree-counting problem:
$V(x) = \sum_{n\geq 0} v_n \frac{x^n}{n!}$ where v_n = # vertex-rooted trees on $[n]$ $= n \cdot t_n = n \cdot$ # trees on $[n]$

has $V(x) = A^{\langle -1\rangle}(x)$ for $A(x) = xe^{-x}$

So Lagrange $\Rightarrow \frac{v_n}{n!} = [x^n]\,V(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{xe^{-x}}\right)^n = \frac{1}{n}[x^{n-1}]\,e^{nx} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}$

$\Rightarrow v_n = n^{n-1}$, and hence $\boxed{t_n = \frac{v_n}{n} = n^{n-2}}$ ← Cayley's formula

We'll see another, very different pf. of Cayley's formula later...

For a proof of Lagrange inversion, see
Thm 2.2.1.5 of Ardila

It can be proved w/ standard analysis (calculus), but I am skipping proof for time considerations. (Maybe we'll look at Lagrange inversion more next semester...)

10/18 (Stanley §1.7)

New topic: q-analogs + q-binomial coefficients

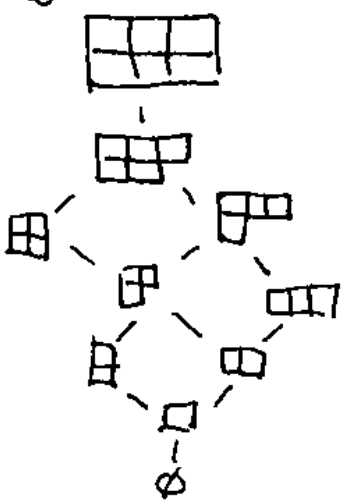
Recall $\displaystyle\sum_{\substack{\text{all partitions}\\ \lambda}} q^{|\lambda|} = \sum_{n\geq 0} p(n)\, q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$

and $\displaystyle\sum_{\lambda:\lambda_1\leq k} q^{|\lambda|} = \sum_{n\geq 0} p_{\leq k}(n)\, q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$

$\lambda \updownarrow \lambda^t$ $\displaystyle \overset{\shortparallel}{\sum_{\lambda:\ell(\lambda)\leq k}} q^{|\lambda|}$

Q: what about $\displaystyle \begin{bmatrix} j+k \\ k \end{bmatrix}_q := \sum_{\substack{\lambda:\lambda_1\leq j\\ \ell(\lambda)\leq k}} q^{|\lambda|}$?

e.g. $j=3,\ k=2$



q^6

q^5

$2q^4 \Rightarrow \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = 1+q+2q^2+2q^3+2q^4+q^5+q^6$

$= (1+q+q^2+q^3+q^4)(1+q^2)$

$2q^3$

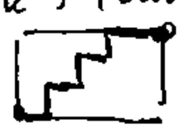
$2q^2$

q

1

We call $\begin{bmatrix} j+k \\ k \end{bmatrix}_q$ the q-binomial coefficient because when $q=1$ it is $=$ binomial coeff. $\binom{j+k}{k}$.

(Recall that $\binom{j+k}{k}$ counts N, E lattice paths $(0,0)\to(j,k)$:



$\Longleftrightarrow$ partitions $\lambda \subseteq j\times k$ rectangle.)

Now let's record properties of $\begin{bmatrix} j+k \\ k \end{bmatrix}_q$:

Prop. (a) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q \overset{q=1}{\rightsquigarrow} \binom{j+k}{k}$ (as we just explained...)

(b) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k \\ j \end{bmatrix}_q$ (since $k \boxed{\lambda} \leftrightarrow \boxed{\lambda^t}\, j$)

(c) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \sum_{n=0}^{jk} p(j,k,n)\, q^n$ has symmetric coefficients: $p(j,k,n) = p(j,k,jk-n)$

(Since $k \boxed{\lambda \;\; \lambda^c}$ have $|\lambda| + |\lambda^c| = jk$) e.g., coeffs's of $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]_q$ are $(1,1,2,2,2,1,1)$ ← palindrome

(d) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q$ (q-Pascal identity)

($k \boxed{\lambda}$ $\overset{j}{=}$ $_{k-1}\{\boxed{\lambda}$ or $k \boxed{\lambda}$ remove 1^{st} column)

$= q^j \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q$ (by symmetry)

(e) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \sum_{\substack{\text{rearrangements} \\ (w_1,\dots,w_{j+k}) = w \text{ of } 0^j 1^k}} q^{\mathrm{inv}(w)}$, where $\mathrm{inv}(w) := \#\{(a,b) : 1 \le a < b \le j+k,\ w_a > w_b\}$ is the # of inversions of word w.

$(\underbrace{00\cdots0}_{j}\underbrace{11\cdots1}_{k})$

e.g. $\mathrm{inv}(0101001011) = 4+3+1+0 = 8$

Can read boundary of λ backwards as $0 =$ west, $1 =$ south to get word w and then $|\lambda| = \mathrm{inv}(w)$

(f) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \#\left\{ \begin{matrix} k\text{-dimensional} \\ \text{subspaces} \end{matrix} \text{ of } (\mathbb{F}_q)^{j+k} \right\}$ if $q = p^d$ is a prime power, so $q = |\mathbb{F}_q|$

← finite field!

(g) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \dfrac{[j+k]_q!}{[j]_q!\,[k]_q!}$ where $[n]_q! := [1]_q [2]_q \cdots [n]_q$ and $[n]_q := 1+q+q^2+\cdots+q^{n-1} = \dfrac{1-q^n}{1-q}$.

e.g. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = \dfrac{[5]_q!}{[3]_q!\,[2]_q!} = \dfrac{[5]_q [4]_q \cancel{[3]_q} \cancel{[2]_q} \cancel{[1]_q}}{\cancel{[3]_q} \cancel{[2]_q} \cancel{[1]_q}\, [2]_q\, \cancel{[1]_q}} = (1+q+q^2+q^3+q^4) \cdot \dfrac{(1+q+q^2+q^3)}{(1+q)}$

$= (1+q+q^2+q^3+q^4) \cdot (1+q^2)$ ✓

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Pf: (a),(b),(c),(d),(e) explained in comments above.

We could prove (f)+(g) by induction + q-Pascal, but instead...

For (f) we claim that there is a bijection:

$\{k\text{-dim'l subspaces } V \subseteq (\mathbb{F}_q)^{j+k}\}$ $\qquad$ Row Space(A)

$\Updownarrow$ See LINEAR ALGEBRA LEMMA below $\qquad \uparrow$

$\left\{ \begin{array}{l} \text{matrices } A \in \mathbb{F}_q^{k \times (j+k)} \text{ of (full)} \\ \text{rank } k \text{ in row-reduced echelon form} \end{array} \right\}$ $\qquad A$

$\downarrow \pi$

$\{\text{Partitions } \lambda \subseteq k \times j\}$

Shape of $*$'s (= nonzero entries in non-pivot cols) read backwards.

$j+k=13,\ j=9$

e.g. $k=4$
$$\begin{bmatrix} 0&0&1&*&*&0&*&0&*&*&*&0&* \\ 0&0&0&0&0&1&*&0&*&*&*&0&* \\ 0&0&0&0&0&0&0&1&*&*&*&0&* \\ 0&0&0&0&0&0&0&0&0&0&0&1&* \end{bmatrix}$$

$j=9$, $k=4$

$$\begin{array}{ccccccc} * & * & * & * & * & * & * \\ & * & * & * & * & * & \\ & & * & * & * & & \\ & & & & * & & \end{array}$$

LINEAR ALGEBRA LEMMA: If $A, B \in \mathbb{F}_q^{k \times (j+k)}$ are both in RREF, and have the same row space, then $A = B$.

Note: $|\pi^{-1}(\lambda)| = q^{|\lambda|}$, since can choose $*$'s from $\mathbb{F}_q$ arbitrarily.

$$\Rightarrow \#\{k\text{-dim'l subspaces } V \subseteq \mathbb{F}_q^{j+k}\} = \sum_{\lambda \subseteq k \times j} \#\pi^{-1}(\lambda) = \sum_{\lambda \subseteq k \times j} q^{|\lambda|} = \begin{bmatrix} j+k \\ k \end{bmatrix}_q. \checkmark$$

For (g), suffices to check $\#\{k\text{-dim'l subspaces } V \subseteq \mathbb{F}_q^{j+k}\} \overset{?}{=} \dfrac{[j+k]_q!}{[k]_q!\,[j]_q!}$

$$= \frac{\#\{\text{ordered bases } (v_1, v_2, \ldots, v_k) \text{ for all } k\text{-dim'l subspaces in } \mathbb{F}_q^{j+k}\}}{\#\{\text{ordered bases } (v_1, v_2, \ldots, v_k) \text{ for one particular } k\text{-subspace}\}}$$

Pick $v_1 \neq 0$, pick $v_2 \notin \mathbb{F}_q v_1$, pick $v_3 \notin \text{Span}\{v_1, v_2\}$

$$= \frac{(q^{j+k}-1)(q^{j+k}-q)(q^{j+k}-q^2)\cdots(q^{j+k}-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\cdots(q^k-q^{k-1})} \overset{?}{=} \frac{[j+k]_q!}{[k]_q!\,[j]_q!} \checkmark$$

pick $v_1 \neq 0$, pick $v_2 \neq \text{Span}\{v_1\}$

$$= \frac{(q^{j+k}-1)(q^{j+k-1}-1)\cdots(q^{j+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} = \frac{[j+k]_q [j+k-1]_q \cdots [j+1]_q}{[k]_q [k-1]_q \cdots [1]_q} \blacksquare$$

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More generally, one can define the <u>q-multinomial</u> <u>coefficient</u>

$$\begin{bmatrix} n \\ k_1, k_2, \ldots, k_\ell \end{bmatrix}_q := \frac{[n]_q!}{[k_1]_q! [k_2]_q! \cdots [k_\ell]_q!} \quad \text{for } \sum_{i=1}^{\ell} k_i = n$$

$\ell = 2$, $(k_1, k_2) = (k, j)$: $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k \\ j \end{bmatrix}_q = \begin{bmatrix} j+k \\ k, j \end{bmatrix}_q$

$\xrightarrow{q=1} \binom{n}{k_1, k_2, \ldots, k_\ell}$ ~ usual multinomial

<u>Prop.</u> (a) $\begin{bmatrix} n \\ k_1, k_2, \ldots, k_\ell \end{bmatrix}_q = \displaystyle\sum_{\substack{\text{rearrangements} \\ w = (w_1, \ldots, w_n) \\ \text{of } k_1\ 1\text{'s}, \\ k_2\ 2\text{'s}, \ldots \\ k_\ell\ \ell\text{'s}}} q^{\mathrm{inv}(w)}$ (in particular, $\begin{bmatrix} n \\ 1, 1, \ldots, 1 \end{bmatrix}_q = [n]_q! = \displaystyle\sum_{w \in S_n} q^{\mathrm{inv}(w)}$)

(b) $\begin{bmatrix} n \\ k_1, \ldots, k_\ell \end{bmatrix}_q = \#\{$ <u>partial</u> <u>flags</u> of subspaces $\{0\} \subset V_{k_1} \subset V_{k_1+k_2} \subset \cdots \subset V_{k_1+k_2+\cdots+k_{\ell-1}} \subset \mathbb{F}_q^n \}$ where $\dim_{\mathbb{F}_q} V_i = i$

((In particular, $[n]_q! = \#\{$ <u>complete</u> flags $\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{F}_q^n \}$)

<u>Pf:</u> For both, use $\begin{bmatrix} n \\ k_1, \ldots, k_\ell \end{bmatrix}_q \overset{\text{easy!}}{=} \underbrace{\begin{bmatrix} n \\ k_1 \end{bmatrix}_q}_{q\text{-binomial}} \cdot \underbrace{\begin{bmatrix} n-k_1 \\ k_2, k_3, \ldots, k_\ell \end{bmatrix}_q}_{\text{smaller } q\text{-multinomial}}$

(Base cases $\ell = 1 \Rightarrow$ trivial, $\ell = 2 \Rightarrow$ already done above...)

and in the inductive step:

- for (a), note that $\mathrm{inv}(w) = \#\{$inversions between 1's and all of 2's, 3's, ..., ℓ's$\}$ $+ \#\{$inversions between 2's, ..., ℓ's$\}$

e.g. $w = 124213241$

$\mathrm{inv}(w) = \mathrm{inv}(122212221)$ ✓
$\quad + \mathrm{inv}(242\ 324)$

- for (b), note that after fixing V_{k_1}, quotient space.

$\{$flags $\{0\} \subset V_{k_1} \subset V_{k_1+k_2} \subset \cdots \subset \mathbb{F}_q^n\} \leftrightarrow \{$flags $\{0\} \subset V_{k_1+k_2}/V_{k_1} \subset V_{k_1+k_2+k_3}/V_{k_1} \subset \cdots \subset \mathbb{F}_q^n / V_{k_1}\}$, $\mathbb{F}_q^n/V_{k_1} \cong \mathbb{F}_q^{n-k_1}$

∎