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Exponential generating functions (Ardila §2.3)

$\mathcal{A}$ = structure one can place on labelled objects like $[n]$

a_n = # of such structures one can place on $[n]$

$\rightsquigarrow A(x) := \sum_{n\geq 0} a_n \frac{x^n}{n!}$ =: the exponential gen. fun. for $\mathcal{A}$.

Prop. • If $\mathcal{C}$ structures are a choice of $\mathcal{A}$- or $\mathcal{B}$-structures then $C(x) = A(x) + B(x)$, ("$\mathcal{C} = \mathcal{A} + \mathcal{B}$")

• If $\mathcal{C}$ structures on $[n]$ are a choice of partition $[n] = S_1 \sqcup S_2$, with an $\mathcal{A}$-structure on S_1, $\mathcal{B}$-structure on S_2, ("$\mathcal{C} = \mathcal{A} * \mathcal{B}$")

so that $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$, then $C(x) = A(x) \cdot B(x)$.

technical pt.: need $a_0 = 0$ here ↪ • If $\mathcal{C}$-structures are a choice of (unordered) set partition π of $[n]$, and then an $\mathcal{A}$-structure on each block of π,

"The exponential formula" → then $\boxed{C(x) = e^{A(x)}}$ ("$\mathcal{C} = \mathrm{Set}(\mathcal{A})$")

Pf: • $\mathcal{C} = \mathcal{A} + \mathcal{B}$ is obvious.

• For $\mathcal{C} = \mathcal{A} * \mathcal{B}$, note $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \Leftrightarrow \frac{c_n}{n!} = \sum_{i+j=n} \frac{a_i}{i!} \frac{b_j}{j!}$

$\Leftrightarrow C(x) = A(x)B(x)$. ✓

• For $\mathcal{C} = \mathrm{Set}(\mathcal{A})$, note $\mathcal{C} = \bigsqcup_{k=0}^{\infty} \mathcal{A}^{(k)}$, where $\mathcal{A}^{(k)}$ = {pick a set partition of $[n]$ into exactly k blocks, put $\mathcal{A}$-structure on each block}

So $C(x) = \sum_{k=0}^{\infty} A^{(k)}(x)$.

But $k!\,A^{(k)}(x) = A(x)^k$ = e.g.f for $\underbrace{\mathcal{A} * \mathcal{A} * \cdots * \mathcal{A}}_{k \text{ times}}$

= {pick a set partition $[n] = B_1 \sqcup \cdots \sqcup B_k$ into k ordered blocks, and put $\mathcal{A}$-structure on each block}

note: $a_0 = 0 \Rightarrow$ all $B_i \neq \emptyset$

Hence, $A^{(k)}(x) = \frac{A(x)^k}{k!}$,

and so $C(x) = \sum_{k=0}^{\infty} \frac{A(x)^k}{k!} = e^{A(x)}$. ✓ ∎

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Examples of e.g.f.'s:

hat-check problem

① Recall $d_n = \#\{\text{derangements in } S_n\}$, $D(x) := \sum_{n\geq 0} \frac{d_n}{n!} x^n$

$$\left\{\begin{array}{c}\text{all}\\ \text{permutations}\end{array}\right\} = \left\{\begin{array}{c}\text{fixed point only perms,}\\ \text{i.e., identity perms}\end{array}\right\} * \left\{\begin{array}{c}\text{derangements,}\\ \text{(fixed-pt-free perms)}\end{array}\right\}$$

So $\sum_{n\geq 0} \cancel{n!} \frac{x^n}{\cancel{n!}} = \left(\sum_{n\geq 0} 1 \cdot \frac{x^n}{n!}\right) \cdot D(x)$

$$\frac{1}{1-x} = e^x \cdot D(x), \text{ i.e., } D(x) = \frac{e^{-x}}{1-x}, \text{ as we saw. } \checkmark$$

② $\left\{\begin{array}{c}\text{involutions}\\ \sigma^2 = 1\end{array}\right\} = \text{Set}\left(\left\{\begin{array}{c}\text{involutions w/}\\ \text{exactly one cycle}\end{array}\right\}\right)$

Hence $\sum_{n\geq 0} \#\left\{\begin{array}{c}\sigma\in S_n:\\ \sigma^2 = 1\end{array}\right\} \frac{x^n}{n!} = e^{\sum_{n\geq 0} \#\{\sigma\in S_n: \sigma^2 = 1, \#\text{cycles}(\sigma) = 1\} \frac{x^n}{n!}}$

$$= e^{0\cdot\frac{x^0}{0!} + 1\cdot\frac{x^1}{1!} + 1\cdot\frac{x^2}{2!} + 0\cdot\frac{x^3}{3!} + 0\cdot\frac{x^4}{4!} + \cdots}$$

$$= e^{x + \frac{x^2}{2}}, \text{ as we saw before. } \checkmark$$

③ More generally, Touchard's Thm follows from exp. formula:

$$\{\text{permutations}\} = \text{Set}(\{\text{permutations w/ exactly one cycle}\})$$

and if we weight σ by $t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots$, wt is multiplicative with respect to this decomposition.

So $\sum_{n\geq 0} \frac{x^n}{n!} \left(\sum_{\sigma\in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots\right) = e^{\sum_{n\geq 0} \frac{x^n}{n!} \left(\sum_{\substack{\sigma\in S_n:\\ \sigma \text{ has only one cycle}}} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots\right)}$

$$= e^{\sum_{n\geq 1} \frac{x^n}{n!} \cdot t_n \cdot (n-1)!}$$

there are $(n-1)!$ n-cycles in S_n $(1\, \underbrace{a_1\, a_2 \cdots a_{n-1}}_{\text{arbitrary seq.}})$

$$= e^{\sum_{n\geq 1} t_n \frac{x^n}{n}}$$

$$= e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots}, \text{ as we saw. } \checkmark$$

In addition to permutations, e.g.f.'s are also useful for set partitions and graphs / trees ...

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④ <u>Bell numbers</u> $B_n := \#\{\text{set partitions of } [n]\}$
+ <u>Bell polynomials</u> $B_n(y) := \sum_{\substack{\text{set partition}\\ \pi \text{ of } [n]}} y^{\#\text{blocks}(\pi)} = \sum_{k=0}^{n} S(n,k)\, y^k.$

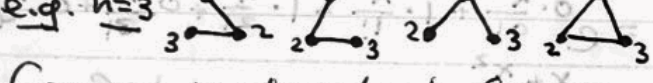
Since $\{\text{set partitions}\} = \text{Set}(\{\text{single (non-empty) block partitions}\})$

$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{1\cdot\frac{x}{1!} + 1\cdot\frac{x^2}{2!} + 1\cdot\frac{x^3}{3!} + \cdots} = e^{(e^x - 1)}$$

and $$\sum_{n\geq 0} B_n(y) \frac{x^n}{n!} = e^{y\cdot\frac{x}{1!} + y\cdot\frac{x^2}{2!} + y\cdot\frac{x^3}{3!} + \cdots} = e^{y(e^x-1)}$$

<u>Cor</u> (extract coeff. of $[y^k]$) $\Rightarrow \sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$

⑤ Let's count <u>connected</u>, <u>simple</u> <u>graphs</u> $G = (V^{[n]}, E \subseteq \binom{[n]}{2})$
weighted by $y^{\#E}$ (number of edges)

<u>e.g.</u> $n=3$  $\Rightarrow 3y^2 + y^3$

Can we understand $\text{Conn}(x,y) := \sum_{n\geq 1} \frac{x^n}{n!} \sum_{\substack{\text{connected}\\ \text{graphs } G \text{ on } [n]}} y^{\#E}$?

<u>Note</u>: $\{\text{all simple graphs}\} = \text{Set}(\{\text{connected simple graphs}\})$

So $\text{All}(x,y) = e^{\text{Conn}(x,y)}$

$$\Rightarrow \text{Conn}(x,y) = \log(\text{All}(x,y)) = \log\Big(\sum_{n\geq 0} \frac{x^n}{n!} \sum_{\substack{\text{simple}\\ \text{graphs on } [n]}} y^{\#E}\Big)$$

$$= \log\Big(1 + \sum_{n\geq 1} \frac{x^n (1+y)^{\binom{n}{2}}}{n!}\Big)$$ ← include, or not, each edge!

computer

$$= x + \frac{x^2}{2!}\cdot 1\cdot y + \frac{x^3}{3!}(3y^2+y^3) + \frac{x^4}{4!}(16y^3+15y^4+6y^5+y^6) + \cdots$$

⑥ Let's try to understand $t_n := \#\{\underline{\text{trees}} \text{ on } [n]\}$

n	trees	t_n
1	·1	1
2	1—2	1
3		3
4	...	16
5	...	125

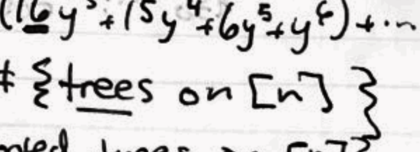
If we set $v_n := \#\{\underline{\text{vertex}}\text{-rooted trees on } [n]\}$

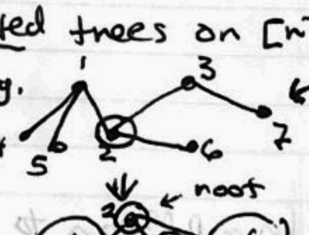
then $v_n = n \cdot t_n$

and $\mathcal{V} = \{\text{root}\} * \text{Set}(\mathcal{V})$

so that $\boxed{V(x) = x e^{V(x)}}$

$V(x) = \sum_{n\geq 0} v_n \frac{x^n}{n!}$

e.g.  ← tree

 ← root ← Set($\mathcal{V}$)

Is $V(x) = xe^{V(x)}$ useful? Yes! Can rephrase as $V(x)e^{-V(x)} = x$, i.e., $V(x)$ is compositional inverse of xe^{-x}.

Prop. If $A(x) = a_1 x + a_2 x^2 + \dots \in R[[x]]$ has zero constant term ($a_0 = 0$), so that $B(A(x))$ is well-defined, then A has a compositional inverse $B = A^{<-1>}$, satisfying $B(A(x)) = A(B(x)) = x \iff a_1 \in R^\times$ is a unit.

But why does knowing $V(x) = A^{<-1>}(x)$ for $A(x) = xe^{x}$ help?

Lagrange inversion thm:
If $B(x) = A^{<-1>}(x)$, that is $B(A(x)) = x$ for some $A(x), B(x) \in \mathbb{C}[[x]]$, then
$$[x^n]\,B(x) = \frac{1}{n}[x^{-1}]\left(\frac{1}{A(x)^n}\right) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{A(x)}\right)^n.$$

Let's see how Lagrange inversions solves tree-counting problem:
$$V(x) = \sum_{n\geq 0} v_n \frac{x^n}{n!} \quad \text{where } v_n = \#\text{ vertex-rooted trees on } [n] = n \cdot t_n = n \cdot \#\text{ trees on } [n]$$
has $V(x) = A^{<-1>}(x)$ for $A(x) = xe^{-x}$

So Lagrange $\Rightarrow$ $\frac{v_n}{n!} = [x^n]V(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{xe^{-x}}\right)^n = \frac{1}{n}[x^{n-1}]e^{nx} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}$

$\Rightarrow v_n = n^{n-1}$, and hence $\boxed{t_n = \frac{v_n}{n} = n^{n-2}}$ ← Cayley's formula

We'll see another, very different pf. of Cayley's formula later...

For a proof of Lagrange inversion, see
Thm 2.2.1.5 of Ardila

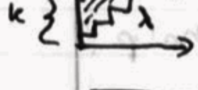
It can be proved w/ standard analysis (calculus), but I am skipping proof for time considerations. (Maybe we'll look at Lagrange inversion more next semester...)

10/18 (Stanley §1.7)

New topic: q-analogs + q-binomial coefficients

Recall $\displaystyle\sum_{\substack{\text{all partitions}\\ \lambda}} q^{|\lambda|} = \sum_{n\geq 0} p(n)\, q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$

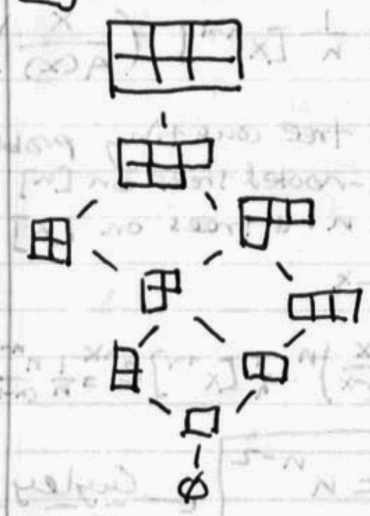
and $\displaystyle\sum_{\lambda:\lambda_1\leq k} q^{|\lambda|} = \sum_{n\geq 0} P_{\leq k}(n)\, q^n = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$



$\displaystyle \overset{\lambda \updownarrow \lambda^t}{=}\ \sum_{\lambda:\ell(\lambda)\leq k} q^{|\lambda|}$

Q: what about $\displaystyle\begin{bmatrix} j+k \\ k \end{bmatrix}_q := \sum_{\substack{\lambda:\lambda_1\leq j\\ \ell(\lambda)\leq k}} q^{|\lambda|}$?

e.g. $j=3$, $k=2$



q^6

q^5

$2q^4$ $\Rightarrow \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = 1+q+2q^2+2q^3+2q^4+q^5+q^6$

$= (1+q+q^2+q^3+q^4)(1+q^2)$

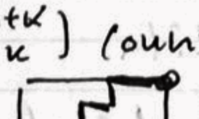
$2q^3$

$2q^2$

q

1

We call $\begin{bmatrix} j+k \\ k \end{bmatrix}_q$ the q-binomial coefficient because when $q=1$ it is $=$ binomial coeff. $\binom{j+k}{k}$.

(Recall that $\binom{j+k}{k}$ counts N, E lattice paths $(0,0)\to(j,k)$:  $\Longleftrightarrow$ partitions $\lambda \subseteq j\times k$ rectangle.)

Now let's record properties of $\begin{bmatrix} j+k \\ k \end{bmatrix}_q$:

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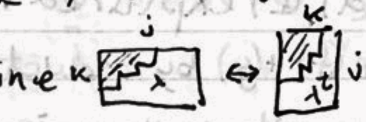
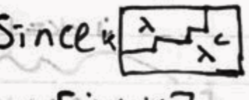
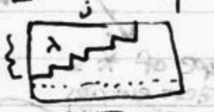
<u>Prop</u>. (a) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q \overset{q=1}{\rightsquigarrow} \binom{j+k}{k}$ (as we just explained...)

(b) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k \\ j \end{bmatrix}_q$ (since )

(c) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \sum_{n=0}^{jk} p(j,k,n) q^n$ has <u>symmetric</u> <u>coefficients</u>:
$p(j,k,n) = p(j,k,jk-n)$

(Since  have $|\lambda|+|\lambda^c| = jk$) e.g., coeff's of $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]_q$ are
$(1,1,2,2,2,1,1)$ ← palindrome

(d) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q$ (q-<u>Pascal</u> <u>identity</u>)

( =  or  remove <u>1st</u> <u>column</u>)

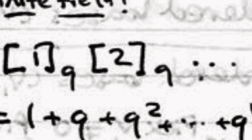
$= q^j \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q$ (by symmetry)

(e) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \sum_{\substack{\text{rearrangements} \\ (w_1,\dots,w_{j+k})=w \text{ of } 0^j 1^k}} q^{\mathrm{inv}(w)}$, where $\mathrm{inv}(w) = \#\{(a,b) : 1 \le a < b \le j+k,\ w_a > w_b\}$ is the # of <u>inversions</u> of word w.

$(\underbrace{00\cdots0}_{j}\underbrace{11\cdots1}_{k})$

Can read boundary of λ backwards as
$0 = $ west
$1 = $ south
to get word w ⟹
and then $|\lambda| = \mathrm{inv}(w)$

e.g. $\mathrm{inv}(01010010\,1) = 4+3+1+0 = 8$



(f) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \#\left\{\begin{matrix}k\text{-}\underline{\text{dimensional}} \\ \underline{\text{subspaces}}\end{matrix} \text{ of } (\mathbb{F}_q)^{j+k}\right\}$ if $q = p^d$ is a <u>prime</u> <u>power</u>,
so $q = |\mathbb{F}_q|$
↖ <u>finite</u> <u>field</u>!

(g) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \dfrac{[j+k]_q!}{[j]_q!\,[k]_q!}$ where $[n]_q! := [1]_q [2]_q \cdots [n]_q$
and $[n]_q := 1+q+q^2+\cdots+q^{n-1} = \dfrac{1-q^n}{1-q}$.

<u>e.g.</u>, $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = \dfrac{[5]_q!}{[3]_q!\,[2]_q!} = \dfrac{[5]_q [4]_q \cancel{[3]_q} \cancel{[2]_q} \cancel{[1]_q}}{\cancel{[3]_q} \cancel{[2]_q} \cancel{[1]_q} [2]_q \cancel{[1]_q}} = (1+q+q^2+q^3+q^4) \cdot \dfrac{(1+q+q^2+q^3)}{(1+q)}$ ✓

$= (1+q+q^2+q^3+q^4) \cdot (1+q^2)$.

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Pf: (a), (b), (c), (d), (e) explained in comments above.

We could prove (f) + (g) by induction + q-Pascal, but instead...

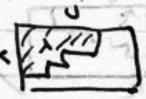
For (f) we claim that there is a bijection:

$\{k\text{-dim'l subspaces } V \subseteq (\mathbb{F}_q)^{j+k}\}$ $\qquad$ Row Space(A)

$\Updownarrow$ see LINEAR ALGEBRA LEMMA below $\qquad \uparrow$

$\{$matrices $A \in \mathbb{F}_q^{k\times(j+k)}$ of (full) rank k in row-reduced echelon form$\}$ $\qquad A$

$\downarrow \pi$

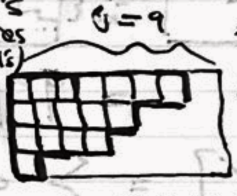
$\{$Partitions $\lambda \subseteq k \times j$  $\}$

$j+k=13,\ j=9$

e.g. $k=4$

$$\begin{bmatrix} 0&0&1&*&*&0&*&0&*&*&*&0&* \\ 0&0&0&0&0&1&*&0&*&*&*&0&* \\ 0&0&0&0&0&0&0&1&*&*&*&0&* \\ 0&0&0&0&0&0&0&0&0&0&0&1&* \end{bmatrix}$$

Shape of $*$'s (= nonzero entries in non-pivot col's) read backwards: $j=9$, $k=4$



$$\begin{matrix} *&*&*&*&*&*&* \\ &&*&*&*&*&* \\ &&&*&*&*&* \\ &&&&&&* \end{matrix}$$

LINEAR ALGEBRA LEMMA: If $A, B \in \mathbb{F}_q^{k\times(j+k)}$ are both in RREF, and have the same row space, then $A = B$.

Note: $|\pi^{-1}(\lambda)| = q^{|\lambda|}$, since can choose $*$'s from $\mathbb{F}_q$ arbitrarily.

$$\Rightarrow \#\{k\text{-dim'l subspaces } V \subseteq \mathbb{F}_q^{j+k}\} = \sum_{\lambda \subseteq k\times j} \#\pi^{-1}(\lambda) = \sum_{\lambda\subseteq k\times j} q^{|\lambda|} = \begin{bmatrix} j+k \\ k\end{bmatrix}_q. \checkmark$$

For (g), suffices to check $\#\{k\text{-dim'l subspaces } V \subseteq \mathbb{F}_q^{j+k}\} \overset{?}{=} \dfrac{[j+k]_q!}{[k]_q!\,[j]_q!}$

$$\frac{\#\{\text{ordered bases } (v_1, v_2, \dots, v_k) \text{ for all } k\text{-dim'l subspaces in } \mathbb{F}_q^{j+k}\}}{\#\{\text{ordered bases } (v_1, v_2, \dots, v_k) \text{ for one particular } k\text{-subspace}\}}$$

Pick $v_1 \neq 0$, pick $v_2 \notin \mathbb{F}_q v_1$, pick $v_3 \notin \mathrm{Span}\{v_1, v_2\}$

$$= \frac{(q^{j+k}-1)(q^{j+k}-q)(q^{j+k}-q^2)\cdots(q^{j+k}-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\cdots(q^k-q^{k-1})} = \frac{[j+k]_q!}{[k]_q!\,[j]_q!} \checkmark$$

pick $v_1 \neq 0$, pick $v_2 \notin \mathrm{Span}\{v_1\}$

$$= \frac{(q^{j+k}-1)(q^{j+k-1}-1)\cdots(q^{j+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} = \frac{[j+k]_q[j+k-1]_q\cdots[j+1]_q}{[k]_q[k-1]_q\cdots[1]_q} \blacksquare$$

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More generally, one can define the q-multinomial coefficient

$$\begin{bmatrix} n \\ k_1,k_2,\dots,k_\ell\end{bmatrix}_q := \frac{[n]_q!}{[k_1]_q!\,[k_2]_q!\cdots[k_\ell]_q!} \quad \text{for } \sum_{i=1}^{\ell} k_i = n$$

$\ell=2$, $(k_1,k_2)=(k,j)$: $\begin{bmatrix} j+k \\ k\end{bmatrix}_q = \begin{bmatrix} j+k \\ j\end{bmatrix}_q = \begin{bmatrix} j+k \\ k,j\end{bmatrix}_q$ $\xrightarrow{q=1}$ $\binom{n}{k_1,k_2,\dots,k_\ell}$ usual multinomial

Prop. (a) $$\begin{bmatrix} n \\ k_1,k_2,\dots,k_\ell\end{bmatrix}_q = \sum_{\substack{\text{rearrangements } w=(w_1,\dots,w_n) \\ \text{of } k_1\ 1\text{'s},\ k_2\ 2\text{'s}, \dots,\ k_\ell\ \ell\text{'s}}} q^{\mathrm{inv}(w)}$$
(In particular, $\begin{bmatrix} n \\ 1,1,\dots,1\end{bmatrix}_q = [n]_q! = \sum_{w\in S_n} q^{\mathrm{inv}(w)}$)

(b) $\begin{bmatrix} n \\ k_1,\dots,k_\ell\end{bmatrix}_q = \#\{$ partial flags of subspaces $\{0\}\subset V_{k_1}\subset V_{k_1+k_2}\subset\cdots\subset V_{k_1+k_2+\cdots+k_{\ell-1}}\subset \mathbb{F}_q^n\}$ where $\dim_{\mathbb{F}_q} V_i = i$

(In particular, $[n]_q! = \#\{$complete flags $\{0\}\subset V_1\subset V_2\subset\cdots\subset V_{n-1}\subset\mathbb{F}_q^n\}$)

Pf: for both, use $\begin{bmatrix} n \\ k_1,\dots,k_\ell\end{bmatrix}_q \overset{\text{easy!}}{=} \underbrace{\begin{bmatrix} n \\ k_1\end{bmatrix}_q}_{q\text{-binomial}} \cdot \underbrace{\begin{bmatrix} n-k_1 \\ k_2,k_3,\dots,k_\ell\end{bmatrix}_q}_{\text{smaller } q\text{-multinomial}}$

(Base cases $\ell=1 \Rightarrow$ trivial, $\ell=2\Rightarrow$ already done above...)
and in the inductive step:

- for (a), note that $\mathrm{inv}(w) = \#\{$inversions between 1's and all of 2's, 3's, ..., ℓ's$\}$ $+\ \#\{$inversions between 2's, ..., ℓ's$\}$

e.g. $w = 124\,213241$

$\mathrm{inv}(w) = \mathrm{inv}(122212221)$ ✓
$\quad + \mathrm{inv}(242\ 324)$

- for (b), note that after fixing V_{k_1},

$\{$flags $\{0\}\subset V_{k_1}\subset V_{k_1+k_2}\subset\cdots\subset\mathbb{F}_q^n\}\leftrightarrow\{$flags $\{0\}\subset V_{k_1+k_2}/V_{k_1}\subset V_{k_1+k_2+k_3}/V_{k_1}\subset\cdots\subset \mathbb{F}_q^n/V_{k_1}\cong\mathbb{F}_q^{n-k_1}\}$ (quotient space) ∎