Descents of permutations

DEF'N For $w = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ w_1 & w_2 & w_3 & \cdots & w_n \end{pmatrix} \in S_n$,

its descent set $D(w) := \{ i : 1 \le i \le n-1,\ w_i > w_{i+1} \}$

$des(w) := \# D(w)$ descent number

$maj(w) := \sum_{i \in D(w)} i$ major index (considered by MacMahon)

Also recall inversion set $I(w) := \{ (i,j) : 1 \le i < j \le n,\ w_i > w_j \}$

and $inv(w) := \# I(w)$ inversion number

We just saw $\sum_{w \in S_n} q^{inv(w)} = [n]_q!$. What about...

Eulerian polynomial $A_n(x) := \sum_{w \in S_n} x^{1+des(w)}$

Mahonian polynomial $Mahon(q) := \sum_{w \in S_n} q^{maj(w)}$

E.g. $n=1$: $A_1(x) = x^1 = x$

$Mahon(q) = q^0 = 1 = [1]_q!$

$n=2$: $A_2(x) = \underset{12}{x^1} + \underset{21}{x^2}$

$Mahon(q) = q^0 + q^1 = 1+q = [2]_q!$

$n=3$ w	$des(w)$	$maj(w)$	$inv(w)$
123	0	0	0
132	1	2	1
213	1	1	1
231	1	2	2
312	1	1	2
321	2	3	3

$A_3(x) = x + 4x^2 + x^3$

$Mahon(q) = 1 + 2q + 2q^2 + q^3$

$= (1+q)(1+q+q^2)$

$= [3]_q! = \sum_{w \in S_3} q^{inv(w)}$

$n=4$: $A_4(x) = x + 11x^2 + 11x^3 + x^4$, $Mahon(q) = [4]_q!$

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Thm 1 $\mathrm{Mahon}(q) = [n]_q!$

Rmk: [Stanley, §1.4] gives bijective pf that $\sum_{w \in S_n} q^{\mathrm{inv}(w)} = \sum_{w \in S_n} q^{\mathrm{maj}(w)}$

Thm 2 $\sum_{m \geq 0} m^n x^m = \dfrac{A_n(x)}{(1-x)^{n+1}}$

$\quad\quad \| $

$(x \cdot d/dx)^n \left(\frac{1}{1-x}\right)$ ~ the way Euler thought about these polynomials

e.g. $(x \cdot d/dx)\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2} = \frac{A_1(x)}{(1-x)^2}$

$(x \cdot d/dx)^2 \left(\frac{1}{1-x}\right) = (x \cdot d/dx)\left(\frac{x}{(1-x)^2}\right) = \frac{x^2+x}{(1-x)^3} = \frac{A_2(x)}{(1-x)^3}$

$\quad\quad \| $

$(x \cdot d/dx)^2 \sum_{m \geq 0} x^m = x\, d/dx\; x\, d/dx \sum_{m \geq 0} x^m = \sum_{m \geq 0} m^2 x^m$. ✓

Let's deduce these from...

Thm (a) $\left(\frac{1}{1-q}\right)^n = \dfrac{\sum_{w \in S_n} q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)\cdots(1-q^n)}$ ($\Rightarrow$ Thm 1 by clearing denominator)

(b) $\sum_{m \geq 0} ([m]_q)^n x^m = \dfrac{\sum_{w \in S_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)}}{(1-x)(1-xq)(1-xq^2)\cdots(1-xq^n)}$ ($\Rightarrow$ Thm 2 by $\lim q \to 1$)

Proof: For (a), note that

$$\mathrm{LHS} = \left(\frac{1}{1-q}\right)^n = \sum_{\substack{f: [n] \to \mathbb{N} \\ "(f_1, f_2, \ldots, f_n)}} q^{\overbrace{f_1 + f_2 + \cdots + f_n}^{|f|}}$$

(simple)

Lemma Every $f: [n] \to \mathbb{N}$ has a unique permutation $w \in S_n$ such that f is w-compatible in the sense that

- $f_{w_1} \geq f_{w_2} \geq \cdots \geq f_{w_n}$
- and $f_{w_i} > f_{w_{i+1}}$ if $i \in D(w)$ (i.e., if $w_i > w_{i+1}$)

<u>Pf of lemma</u>:

e.g. $f = \overset{1}{(2}, \overset{2}{0}, \overset{3}{5}, \overset{4}{0}, \overset{5}{3}, \overset{6}{3}, \overset{7}{2}, \overset{8}{0})$ has $\overset{5}{f_3} \ge \overset{3}{f_5} \ge \overset{3}{f_6} > \overset{2}{f_1} \ge \overset{2}{f_7} > \overset{0}{f_2} \ge \overset{0}{f_4} \ge \overset{0}{f_8}$

So is w-compatible for $w = (3, 5, 6, \cdot 1, 7, \cdot 2, 4, 8) \in S_8$. ■

Thus
$$\text{LHS} = \sum_{w \in S_n} \sum_{\substack{f:[n]\to\mathbb{N}, \\ w\text{-compatible}}} q^{|f|}$$

subtract off the smallest w-compatible f_0 from f to get λ:

$$\begin{array}{l} (\overset{3}{5}, \overset{5}{3}, \overset{6}{3}, \overset{\cdot 1}{2}, \overset{7}{2}, \overset{\cdot 2}{0}, \overset{4}{0}, \overset{8}{0}) = f \\ -(2,2,2,1,1,0,0,0) = f_0 \\ \hline (3,1,1,1,1,0,0,0) = \lambda \end{array}$$

$\boxed{\text{NOTE: } |f_0| = \mathrm{maj}(w) \text{ and } \max(f_0) = \mathrm{des}(w)}$ ■

$$= \sum_{w \in S_n} \sum_{\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)} q^{\mathrm{maj}(w) + |\lambda|}$$

$$= \sum_{w \in S_n} q^{\mathrm{maj}(w)} \sum_{\lambda : \ell(\lambda) \le n} q^{|\lambda|}$$

$$= \sum_{w \in S_n} q^{\mathrm{maj}(w)} \cdot \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

10/27 For (b), we'll do something similar to show

$$(1-x) \sum_{m \ge 0} ([m]_q)^n x^m = \frac{\sum_{w \in S_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)}}{(1-xq)(1-xq^2)\cdots(1-xq^n)}$$

Note,
$$\text{LHS} = (1-x) \sum_{m \ge 0} x^m \sum_{f:[n] \to \{0,1,\ldots,m-1\}} q^{|f|} = (1-x) \sum_{m \ge 0} x^m \sum_{\substack{f:[n]\to\mathbb{N} \\ \max(f) \le m-1}} q^{|f|}$$

Cancellation from $(1-x)$ factor
$$= \sum_{m \ge 0} x^m \sum_{\substack{f:[n]\to\mathbb{N} \\ \max(f) = m-1}} q^{|f|} = \sum_{f:[n]\to\mathbb{N}} x^{\max(f)+1} q^{|f|}$$

$$= \sum_{w \in S_n} \sum_{\substack{f:[n]\to\mathbb{N} \\ w\text{-compatible}}} x^{\max(f)+1} q^{|f|}$$

subtract off the smallest w-compatible f_0 from f to get λ

$$= \sum_{w \in S_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)} \underbrace{\sum_{\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)} x^{\overbrace{\max(\lambda)}^{\lambda_1}} q^{|\lambda|}}$$

same as $\sum_{\lambda : \lambda_1 \le n} x^{\ell(\lambda)} q^{|\lambda|}$ via $\lambda \leftrightarrow \lambda^t$

$$= \sum_{w \in S_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)} \cdot \frac{1}{(1-xq)(1-xq^2)\cdots(1-xq^n)} \checkmark$$

■

New Q: Can we count $\beta(S) := \#\{w \in S_n : D(w) = S\}$
for a subset $S \subseteq [n-1]$?

Or even better, $\beta(S,q) := \sum_{w \in S_n, D(w) = S} q^{inv(w)}$? $\Big\}\ q := 1$

e.g. $n = 4$, $S = \{2\}$

$w: D(w) = S$	$inv(w)$
$13 \cdot 24$	1
$14 \cdot 23$	2
$23 \cdot 14$	2
$24 \cdot 13$	3
$34 \cdot 12$	4

$\Rightarrow \beta(S,q) = q + 2q^2 + q^3 + q^4$
$\downarrow q=1$
$\beta(S) = 5$

10/29 It turns out to be easier to count $\alpha(S) := \#\{w \in S_n : D(w) \subseteq S\}$
and $\alpha(S,q) := \sum_{w \in S_n, D(w) \subseteq S} q^{inv(w)}$

$$\alpha(S,q) = \sum_{\substack{w \in S_n \\ D(w^{-1}) \subseteq S}} q^{inv(w)} \quad (\text{since } inv(w^{-1}) = inv(w))$$

$$= \sum_{\substack{\text{rearrangements} \\ w = (w_1, w_2, \ldots, w_n) \\ \text{of } 1^{k_1}, 2^{k_2}, \ldots, \ell^{k_\ell}}} q^{inv(w)} = \begin{bmatrix} n \\ k_1, k_2, \ldots, k_\ell \end{bmatrix}_q$$

where $\underline{k} = (k_1, k_2, \ldots, k_\ell) \models n$ is the composition for which
S = partial sums $\{k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_{\ell-1}\} \subseteq [n-1]$

because $\{w \in S_n : D(w^{-1}) \subseteq S\}$ = shuffles of $1 < 2 < \cdots < k_1$
$k_1 + 1 < k_1 + 2 < \cdots < k_1 + k_2$ ← inverse descents only occur here
$\vdots$
$k_1 + \cdots + k_{\ell-1} + 1 < \cdots < k_1 + k_2 + \cdots + k_\ell = n$

e.g. $S = \{3, 5\} \subseteq [\overset{n}{\overset{\shortparallel}{8}} - 1]$
$\updownarrow$
$\underline{k} = (3,2,3) \models 8$

rearrangement		shuffle
111 22 333	$\longleftrightarrow$	123 45 678
231 33 211	$\longleftrightarrow$	461 78 523

Note: inverse descents of $w = w_1 w_2 \cdots w_n$
= #'s i s.t. $i+1$ is to left of i
e.g. for these shuffles, can only happen for $i = 3$ or 5 //

So how do we recover $\beta(S)$, $\beta(S,q)$ from $\alpha(S)=\sum_{T\subseteq S}\beta(T)$, $\alpha(S,q)$?

<u>Prop.</u> (Principle of Inclusion-Exclusion)

Given two functions $f_{\subseteq}, f_{=} : 2^{[n]} \longrightarrow R$ (subsets of $[n]$ on $2^{[n]}$; R any abelian group)

$S \longmapsto f_{\subseteq}(S)$

$S \longmapsto f_{=}(S)$

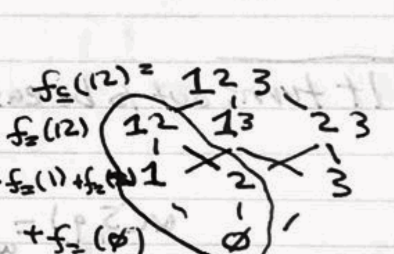
then $f_{\subseteq}(S) \overset{(*)}{=} \sum_{T\subseteq S} f_{=}(T)\ \forall\ S\subseteq [n]$,

$\Leftrightarrow f_{=}(S) = \sum_{T\subseteq S} (-1)^{\#S\setminus T} f_{\subseteq}(T)\ \forall\ S \subseteq [n]$.

<u>e.g.</u> $f_{=}(\emptyset) = f_{\subseteq}(\emptyset)$

$f_{=}(\{i\}) = f_{\subseteq}(\{i\}) - f_{\subseteq}(\emptyset)$

$f_{=}(\{i,j\}) = f_{\subseteq}(\{i,j\}) - f_{\subseteq}(\{i\}) - f_{\subseteq}(\{j\}) + f_{\subseteq}(\emptyset)$



<u>Cor</u> Let $f_{\subseteq}(S) := \alpha(S,q) = \sum_{w\in S_n,\ D(w^{-1})\subseteq S} q^{\mathrm{inv}(w)} = \begin{bmatrix} n \\ k_1, \dots, k_\ell \end{bmatrix}_q$

Then $f_{=}(S) = \beta(S,q) = \sum_{w\in S_n,\ D(w^{-1})\subseteq S} q^{\mathrm{inv}(w)} = \sum_{T\subseteq S} \alpha(T,q)\,(-1)^{\#S\setminus T}$

$= \sum_{\substack{\underline{k}'\models n, \\ \text{coarsening } \underline{k}}} (-1)^{\ell(\underline{k})-\ell(\underline{k}')} \begin{bmatrix} n \\ \underline{k}' \end{bmatrix}_q$

<u>e.g.</u> $n=4$, $S=\{2\}$, $\underline{k} = (2,2)$

$\beta(\{2\},q) = \alpha(\{2\},q) - \alpha(\emptyset,q)$

$= \begin{bmatrix} 4 \\ 2,2 \end{bmatrix}_q - \begin{bmatrix} 4 \\ 4 \end{bmatrix}_q = \frac{[4]_q [3]_q}{[2]_q} - 1$

$= (1+q^2)(1+q+q^2) - 1 = \cancel{1} + q + 2q^2 + q^3 + q^4 \cancel{-1}$

$= q + 2q^2 + q^3 + q^4$ ✓

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Proof of P.I.E. Note $\{f_=(S)\}$ determines $\{f_\subseteq(S)\}_{S\subseteq[n]}$ uniquely via (*), and conversely, by induction on $\#S$, since (*) says

$$f_=(S) = f_\subseteq(S) - \underbrace{\sum_{T\subsetneq S} f_\subseteq(T)}_{\text{already determined}}$$

If we define $g(R) := \sum_{T\subseteq R} (-1)^{\#R\setminus T} f_\subseteq(T) \quad \forall\, R\subseteq[n]$,

then fixing some $S\subseteq[n]$, $\sum_{R\subseteq S} g(R) = \sum_{R\subseteq S}\sum_{T\subseteq R} (-1)^{\#R\setminus T} f_\subseteq(T)$

$$= \sum_{T\subseteq S} f_\subseteq(T) \underbrace{\sum_{T\subseteq R\subseteq S} (-1)^{\#R\setminus T}}$$

$$= \sum_{\hat{R} := R\setminus T \subseteq S\setminus T} (-1)^{\#\hat{R}} = \begin{cases} 1 & \text{if } S=T \\ \#S\setminus T = m: \ \sum_{k=0}^{m} (-1)^k \binom{m}{k} = (1+(-1))^m = 0 & \text{if } T\subsetneq S. \end{cases}$$

$$g(S) = f_\subseteq(S) - \sum_{T\subsetneq S} g(T)$$

$$g(S) = f_=(S) \quad \forall\, S$$

■

Similarly, if $f_\supseteq(S) = \sum_{T\supseteq S} f_=(T)$

then $f_=(S) = \sum_{T\supseteq S} (-1)^{\#T\setminus S} f_\supseteq(T)$

and in particular, $f_=(\emptyset) = \sum_{T} (-1)^{\#T} f_\supseteq(T)$.

e.g., if $A_1, A_2, \ldots, A_n \subseteq \mathcal{U}$ are subsets of some universe $\mathcal{U}$,

then letting $f_\supseteq(S) := \#\left(\bigcap_{i\in S} A_i\right) = \#\{u\in\mathcal{U} : u\in A_i \ \forall\, i\in S\}$

then $f_=(S) = \#\{u\in\mathcal{U} : \{i=1,2,\ldots,n : u\in A_i\} = S\}$

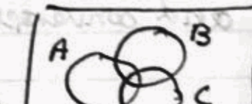
$= \sum_{T\supseteq S} (-1)^{\#T\setminus S} \#\left(\bigcap_{i\in T} A_i\right)$, and in particular

$$\boxed{\#\left(\mathcal{U}\setminus\left(\bigcup_{i=1}^{n} A_i\right)\right) = f_=(\emptyset) = \sum_{T} (-1)^{\#T} \#\left(\bigcap_{i\in T} A_i\right) = \#\mathcal{U} - \sum_{i=1}^{n} \#A_i + \sum_{1\le i<j\le n} \#A_i\cap A_j + \cdots}$$

this is the most common use of P.I.E. ...

Compare to well-known "Venn diagram" formulas:

$\#A\cup B = \#A + \#B - \#A\cap B$



$\#A\cup B\cup C = \#A + \#B + \#C - \#A\cap B - \#A\cap C - \#B\cap C + \#A\cap B\cap C$

Let's see some examples of this formulation of P.I.E.:

(a) Derangements Recall $d_n = \#\{\sigma\in S_n : \sigma \text{ derangement}\}$

Let $A_i := \{\sigma\in S_n : \sigma(i) = i\}$ for $i = 1, 2, \ldots, n$

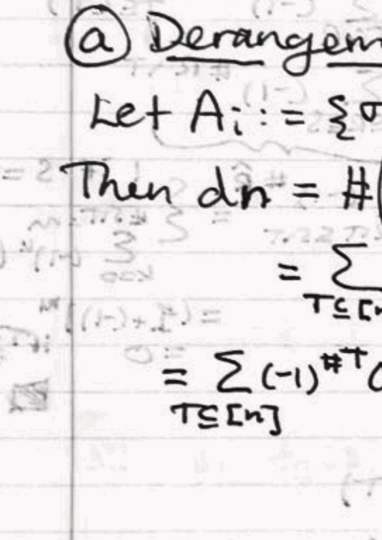
Then $d_n = \#\left(\{\sigma\in S_n\}\setminus \bigcup_{i=1}^{n} A_i\right)$, so by P.I.E. ...

$$= \sum_{T\subseteq[n]} (-1)^{\#T}\, \underbrace{\#\left(\bigcap_{i\in T} A_i\right)}_{\#\{\sigma\in S_n : \sigma(i)=i\ \forall i\in T\} = (n-\#T)!}$$

$$= \sum_{T\subseteq[n]} (-1)^{\#T}(n-\#T)! = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\cdot(n-k)! = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

$$= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

(b) How many N, E lattice paths $(0,0)\to(10,20)$ avoid the points $(2,6)$ and $(7,11)$?



if A_1 = paths that go thru $(2,6)$

A_2 = paths that go thru $(7,11)$

Answer $= \#\left(\underset{(0,0)\to(10,20)}{\text{all paths}} \setminus A_1\cup A_2\right)$

$$= \#\text{all paths} - \#A_1 - \#A_2 + \#A_1\cap A_2$$

$$\#\text{all paths} = \binom{10+20}{10},\quad \#A_1 = \underset{0\to(2,6)}{\binom{2+6}{2}}\cdot\underset{(2,6)\to(10,20)}{\binom{(10-2)+(20-6)}{10-2}},\quad \#A_2 = \binom{7+11}{7}\cdot\binom{3+9}{3},\quad \#A_1\cap A_2 = \underset{(0,0)\to(2,6)}{\binom{2+6}{2}}\underset{(2,6)\to(7,11)}{\binom{5+5}{5}}\underset{(7,11)\to(10,20)}{\binom{3+9}{3}}$$

$$= \binom{30}{10} - \binom{8}{2}\cdot\binom{22}{8} - \binom{18}{7}\cdot\binom{12}{3} + \binom{8}{2}\cdot\binom{10}{5}\cdot\binom{12}{3}$$