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Sign-reversing involutions + identities involving signs

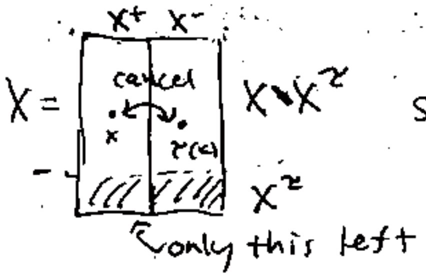
Some identities w/ ± signs can be proven like this:

Prop. Given a set X with a sign function $\mathrm{sgn}: X \to \{\pm 1\}$
a weight function $\mathrm{wt}: X \to R$ (↑ abelian gp.)

and a sign-reversing, weight-preserving, involution $\tau: X \to X$,
$\left(\begin{matrix}\mathrm{sgn}(\tau(x)) = -\mathrm{sgn}(x) \\ \text{if } \tau(x) \neq x\end{matrix}\right)$ $(\mathrm{wt}(\tau(x)) = \mathrm{wt}(x))$ $(\tau^2 = \mathrm{id})$

then $$\sum_{x \in X} \mathrm{sgn}(x) \cdot \mathrm{wt}(x) = \sum_{x \in X^\tau := \{x \in X : \tau(x) = x\}} \mathrm{sgn}(x) \cdot \mathrm{wt}(x).$$

Proof:



$$\mathrm{sgn}(x) \cdot \mathrm{wt}(x) + \underbrace{\mathrm{sgn}(\tau(x))}_{-\mathrm{sgn}(x)} \cdot \underbrace{\mathrm{wt}(\tau(x))}_{\mathrm{wt}(x)} = 0$$

for all $x \in X \setminus X^\tau$ ▨

Examples

① (Warm-up) $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$ for $n \geq 1$

$\parallel$

$\sum_{\text{subsets } S \subseteq [n]} (-1)^{\# S}$

$\mathrm{sgn}: X = 2^{[n]} \to \{\pm 1\}$
$S \mapsto (-1)^{\# S}$

$\mathrm{wt}: X = 2^{[n]} \to \mathbb{Z}$
$S \mapsto 1$

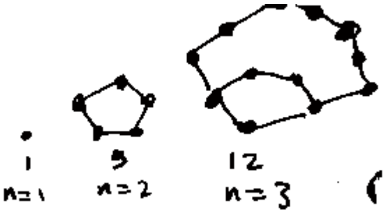
$\tau: \overset{2^{[n]}}{\overset{\parallel}{X}} \to X$

$S \mapsto \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \cup \{1\} & \text{if } 1 \notin S \end{cases}$

is sign-reversing, weight-preserving with no fixed points.

Rmk: This was key identity in pf. of P.I.E.

Pentagonal numbers:



② Recall Thm (Euler's "Pentagonal Number Theorem")

$$\prod_{j\geq 1}(1-q^j) = 1 + \sum_{n\geq 1}(-1)^n\left(q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}}\right)$$

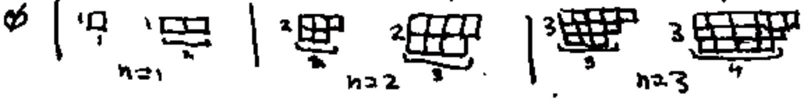
$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots$$

Franklin's (1881) Proof of Euler's P.N.T.:

$$\underline{LHS} = (1-q)(1-q^2)\cdots = \sum_{\substack{\text{partitions}\\ \lambda=\lambda_1>\lambda_2>\cdots>\lambda_\ell>0\\ \text{into distinct parts}}} \overset{\text{sgn}(x)}{(-1)^{\ell(\lambda)}}\ \overset{\text{wt}(x)}{q^{|\lambda|}}$$

$X :=$ partitions $\lambda=\lambda_1>\lambda_2>\cdots>\lambda_\ell>0$ into distinct parts

$$\underline{RHS} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots$$

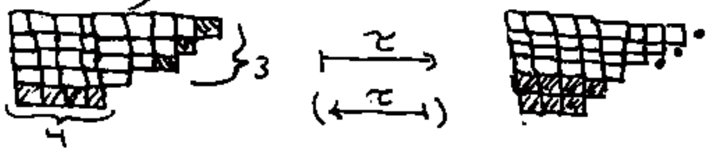


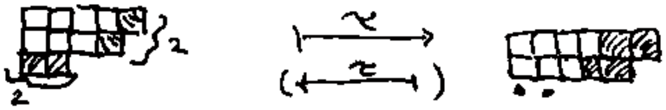
Franklin defined $\tau : X := \{\lambda \text{ w/ distinct parts}\} \to X$ by comparing

- smallest part and
- longest initial run $\lambda_1, \lambda_1-1, \lambda_1-2, \ldots$

and moving the smaller one onto the bigger one:

(or smallest part onto longest run if they are the same size)





involution ↙ sign-reversing ↙ wt-preserving

When one can do this, check $\tau^2 = id$, $\ell(\tau(\lambda)) = \ell(\lambda) \pm 1$, $|\lambda| = |\tau(\lambda)|$

One cannot do this if:

- smallest part + run have the same size and overlap

$n=3$, $|\lambda| = \frac{3n(n-1)}{2}$

- or run is 1 smaller + they overlap

$n=3$, $|\lambda| = \frac{3n(n+1)}{2}$

So τ implies only these shapes contribute to LHS $\Rightarrow$ LHS = RHS. ∎

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③ <u>Theorem</u> (Kirchoff's Matrix-Tree Theorem)

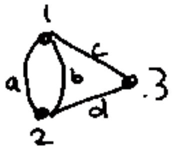
The number of <u>spanning</u> <u>trees</u> in a <u>multigraph</u> $G = (\overset{\text{"}[n]=\{1,\dots,n\}\text{"}}{V}, E)$
(multiple/parallel edges allowed)

for any i $\xrightarrow{\quad}$ is $\det\left(\overline{L(G)}^{i,i}\right)$, where $\overline{A}^{i,i}$ means A w/ row + column i removed,

and $L(G)$ is the $n \times n$ <u>Laplacian</u> <u>matrix</u> of G:

$$L(G)_{v,w} := \begin{cases} \deg(v) & \text{if } v = w \\ -\#\text{edges from } v \text{ to } w & \text{if } v \neq w \end{cases}$$

<u>Note</u>: A <u>spanning</u> <u>tree</u> T of G is a subgraph of G that's a tree and which contains all the vertices V.

<u>Example</u> $G =$ 

has 5 spanning trees:

and $L(G) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 3 & -2 & -1 \\ 2 & -2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{array}$

so $\det\left(\overline{L(G)}^{1,1}\right) = \det\left(\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\right) = 6 - 1 = 5$ ✓

and $\det\left(\overline{L(G)}^{3,3}\right) = \det\left(\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}\right) = 9 - 4 = 5$ ✓

<u>Example</u> Recall Cayley's formula n^{n-2} for # of (labeled) trees on n vertices. These are the ...

Spanning trees of the complete graph K_n on $[n]$.

e.g. $n=5$  $\overline{L(K_n)}^{n,n} = \begin{array}{c|cccc} & 1 & 2 & \cdots & n \\ \hline 1 & n-1 & -1 & -1 \cdots & \\ 2 & -1 & n-1 & -1 \cdots & \\ \vdots & \vdots & & \ddots & \\ n & -1 & & & \end{array} = n \underbrace{I_{n-1}}_{\substack{(n-1)\times(n-1) \\ \text{identity matrix}}} - \underbrace{\mathbb{1}_{n-1}}_{\substack{\text{all 1's matrix} \\ \left[\begin{smallmatrix}1 & \cdots & 1 \\ 1 & \cdots & 1\end{smallmatrix}\right]}}$

What are eigenvalues of $\mathbb{1}_{n-1}$? It has rank $=1$, so $(n-2)$ eigenvalues $=0$

Also $\mathbb{1}_{n-1} \cdot \left[\begin{smallmatrix}1\\ \vdots \\ 1\end{smallmatrix}\right] = (n-1)\left[\begin{smallmatrix}1\\ \vdots \\ 1\end{smallmatrix}\right]$, so one eigenvalue $=(n-1)$.

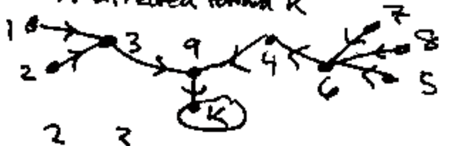
$\mathbb{1}_{n-1}$ has eigenval's $(\overbrace{0,0,\ldots,0}^{n-2},n-1) \Rightarrow \overline{L(K)}^{n,n}$ has eigenval's $(\overbrace{n,n,\ldots,n}^{n-2},1)$

$\Rightarrow \det(\overline{L(K)}^{n,n}) = n^{n-2} \Rightarrow$ Cayley's formula. ✓

In fact, let's prove a weighted, directed version of Kirchoff:

Thm If $L = \begin{array}{c|cccc} & 1 & 2 & \cdots & n \\ \hline 1 & * & -a_{12} & -a_{13} \cdots & -a_{1n} \\ 2 & -a_{21} & * & -a_{23} \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & & \\ n & -a_{n1} & \cdots & \cdots & * \end{array}$ has $L_{ij} = \begin{cases} -a_{ij} & \text{if } i \neq j \\ a_{i1}+a_{i2}+\cdots+\widehat{a_{ii}}+\cdots+a_{in} & \text{if } i=j \end{cases}$ [redacted]

then $\det(\overline{L}^{k,k}) = \sum_{\substack{\text{arborescences} \\ A \text{ directed toward } K}} \prod_{\text{arcs } i \to j \text{ in } A} a_{ij}$, where a_{ij} are formal parameters.

e.g. 

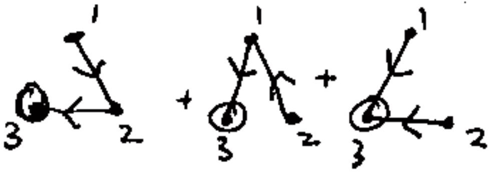
Note: $\Rightarrow$ Kirchoff by setting $a_{ij} = \#$ edges $i \to j$ in G

e.g. $n=3$

$L = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & a_{12}+a_{13} & -a_{12} & -a_{13} \\ 2 & -a_{21} & a_{21}+a_{23} & -a_{23} \\ 3 & -a_{31} & -a_{32} & a_{31}+a_{32} \end{array} \Rightarrow \det(\overline{L}^{3,3}) = \det \begin{bmatrix} a_{12}+a_{13} & -a_{12} \\ -a_{21} & a_{21}+a_{23} \end{bmatrix}$

$= (a_{12}+a_{13})(a_{21}+a_{23}) - (-a_{12})(-a_{21})$

$= a_{12}a_{23} + \cancel{a_{12}a_{21}} + a_{13}a_{21} + a_{13}a_{23} - \cancel{a_{12}a_{21}}$



$a_{12}a_{23} \quad a_{13}a_{21} \quad a_{13}a_{23} = a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{23}$ ✓ $\|$

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Proof of Thm:

$$L = \begin{bmatrix} R_1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & R_2 - a_{22} & \cdots & \\ \vdots & & \ddots & \\ -a_{n1} & & \cdots & R_n - a_{nn} \end{bmatrix} \text{ where } R_i = a_{i1} + a_{i2} + \cdots + a_{ii} + \cdots + a_{in} = \sum_{j=1}^{n} a_{ij}$$

$$= (R_i \cdot \delta_{ij} - a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$$

$$\boxed{\text{Recall: } sgn(w) = (-1)^{inv(w)}}$$

$$\Rightarrow \det(\overline{L}^{n,n}) = \sum_{w \in S_{n-1}} sgn(w) \prod_{i=1}^{n-1} L_{i,w(i)}$$

"Leibniz formula"

$$= \sum_{\substack{S \subseteq [n-1] \\ (\text{fixed by } w)}} \prod_{i \in S} (R_i - a_{ii}) \sum_{\substack{w \in S_{[n-1]\setminus S} \\ \text{a derangement}}} sgn(w) \prod_{i \in [n-1]\setminus S} (-a_{i,w(i)})$$

$$= \sum_{S \subseteq [n-1]} \sum_{T \subseteq S} \prod_{i \in T} R_i \prod_{i \in S \setminus T} (-a_{ii}) \sum_{\substack{w \in S_{[n-1]\setminus S} \\ \text{derang.}}} sgn(w) \cdot \prod_{i \in [n-1]\setminus S} (-a_{i,w(i)})$$

$$\boxed{\begin{array}{l} \text{Important property of sign of permutation:} \\ sgn(v \cdot w) = sgn(v) \cdot sgn(w) \end{array}}$$

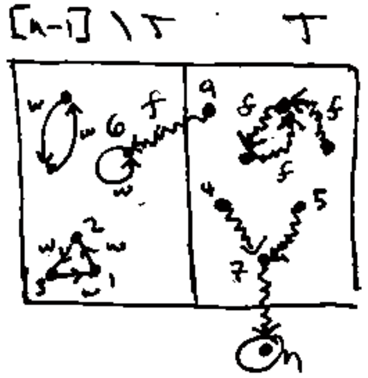
$$= \sum_{T \subseteq [n-1]} \prod_{i \in T} \underbrace{(a_{i1} + a_{i2} + \cdots + a_{in})}_{\substack{\parallel \\ \sum_{f: T \to [n]} \prod_{i \in T} a_{i,f(i)}}} \cdot \sum_{w \in S_{[n-1]\setminus T}} sgn(w) \prod_{i \in [n-1]\setminus T} (-a_{i,w(i)})$$

$$= \sum_{(T,f,w)} \underbrace{(-1)^{\#[n-1]\setminus T} \cdot sgn(w)}_{sgn(x)} \cdot \underbrace{\prod_{i \in T} a_{i,f(i)} \prod_{i \in [n-1]\setminus T} a_{i,w(i)}}_{wt(x)}$$

$$\mathcal{X} := \left\{ \begin{array}{l} (T,f,w) \\ T \subseteq [n-1] \\ f: T \to [n] \\ w \in S_{[n-1]\setminus T} \end{array} \right.$$

We will evaluate this signed, weighted sum using a sign-reversing involution ...

Picture of (T, f, w):

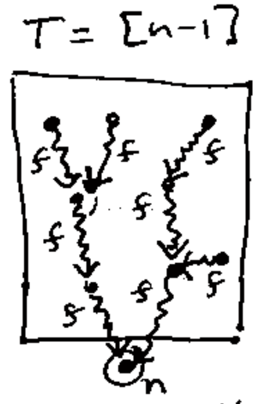


We can define an involution $\tau: \mathcal{X} \rightarrow \mathcal{X}$, that swaps the cycle containing the smallest index $i \in [n-1]$ from w to f or back from f to w!

Check that τ
- is an involution (clear)
- is wt-preserving (preserves arcs)
- is sign-reversing (sign of a k-cycle is $(-1)^{k+1}$ ✓)

What are the fixed points $\mathcal{X}^{\tau}$?

No cycles in w or f $\Rightarrow$ $[n-1] \setminus T$ is empty, i.e., $T = [n-1]$ and $f: [n-1] \rightarrow [n]$ has no cycles



(easy) Lemma. This forces f to be an arborescence directed toward n (and conversely, any arborescence is such an f).

$$\text{Hence, } \det(L^{n,n}) = \sum_{\substack{\text{arborescences} \\ f \text{ on } [n] \text{ directed} \\ \text{toward } n}} \prod_{i \in [n-1]} a_{i,f(i)}.$$

▨

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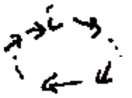
The transfer matrix method (Stanley §4.7, Ardila §3.1.2)

Another tool from linear algebra for counting walks in graphs

Thm Let G be a graph w/ vertex set $V=[n]$, and let $A_G=(a_{ij})_{j=1,\dots,n}^{i=1,\dots,n}$ be its adjacency matrix: $a_{ij}=\#\text{ edges from } i \text{ to } j$.

Then (a) # of walks of length ℓ $i=i_0\to i_1\to\cdots\to i_\ell=j$ $=(A_G^\ell)_{i,j}$ for all $\ell\geq 0$.

(b) # of closed walks of length ℓ $i=i_0\to\cdots\to i_\ell=i$ (for ~~any~~ all i) $=\lambda_1^\ell+\lambda_2^\ell+\cdots+\lambda_n^\ell$,



where $\lambda_1,\lambda_2,\dots,\lambda_n$ are the eigenvalues of A_G.

Pf: (a) is just definition of matrix multiplication:

$$(A_G^\ell)_{i,j}=\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\cdots\sum_{i_{\ell-1}=1}^{n}a_{i,i_1}a_{i_1,i_2}\cdots a_{i_{\ell-1},j}=\text{LHS of (a)}\checkmark$$

For (b), from (a) it follows that

$$\#\text{closed walks of length }\ell=\sum_{i=1}^{n}(A_G^\ell)_{i,i}=\text{trace}(A_G^\ell).$$

Since A_G is real + symmetric, it can be diagonalized, i.e., $\exists P$ s.t. $PA_GP^{-1}=\begin{pmatrix}\lambda_1&&0\\&\ddots&\\0&&\lambda_n\end{pmatrix}$. Thus,

Recall: $\text{tr}(AB)=\text{tr}(BA)$

$$\text{trace}(A_G^\ell)=\text{trace}\left(\left(P^{-1}\begin{pmatrix}\lambda_1&&\\&\ddots&\\&&\lambda_n\end{pmatrix}P\right)^\ell\right)$$

$$=\text{trace}\left(P^{-1}\begin{pmatrix}\lambda_1&&0\\&\ddots&\\0&&\lambda_n\end{pmatrix}^\ell P\right)=\text{trace}\begin{pmatrix}\lambda_1^\ell&&\\&\ddots&\\&&\lambda_n^\ell\end{pmatrix}=\lambda_1^\ell+\cdots+\lambda_n^\ell.\ \square$$

Example Let $f(n,k)=\#$ proper (no adjacent vertices w/ same color) vertex-colorings of C_n ← cycle graph w/ k-colors.

e.g. $n=2$

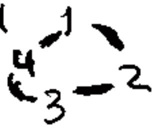
$f(2,k)=k(k-1)$

color 1 first in k ways; color 2 second in $k-1$ ways

$n=3$

$f(3,k)=k(k-1)(k-2)$

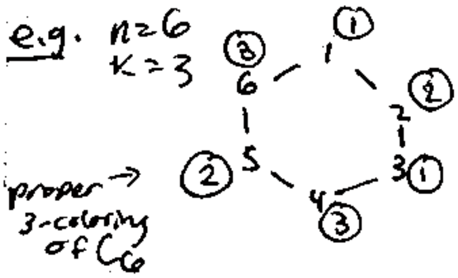
color 1, color 2, color 3

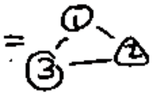
$n=4$ 

$$f(4,k) = \underbrace{\overset{\text{color 1}}{k}\,\overset{\text{color 2}}{(k-1)}\,\overset{\text{color 4}}{(k-2)}\,\overset{\text{color 3}}{(k-2)}}_{\text{2+4 have different colors}} + \underbrace{\overset{\text{color 1}}{k}\,\overset{\text{color 2+4 same}}{(k-1)}\,\overset{\text{color 3}}{(k-1)}}_{\text{2+4 have same color}}$$

Note: $\{\text{proper } k\text{-colorings of } C_n\} \leftrightarrow \left\{\begin{array}{c}\text{closed walks of length } n\\ \text{in complete graph } K_k\end{array}\right\}$

e.g. $n=6$, $k=3$



a closed walk in $K_3 =$  of length 6.

So taking $A_{K_k} = \begin{array}{c|cccc} & ① & ② & \cdots & ⓚ \\ \hline ① & 0 & 1 & \cdots & 1 \\ ② & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & & \vdots \\ ⓚ & 1 & 1 & \cdots & 0 \end{array} = \mathbb{1}_k - I_k,$

which has eigenvalues $(\lambda_1, \ldots, \lambda_k) = (k-1, \overbrace{-1,-1,\ldots,-1}^{k-1 \text{ terms}})$

(since we saw earlier that $\mathbb{1}_k$ has eigen's $(k, \overbrace{0,0,\ldots,0}^{k-1})$)

we find that
$$\begin{aligned} f(n,k) &= \lambda_1^n + \cdots + \lambda_n^n \\ &= (k-1)^n + (-1)^n + \cdots + (-1)^n \\ &= (k-1)^n + (k-1)\ (-1)^n \\ &= (k-1)\left((k-1)^{n-1} + (-1)^n\right). \end{aligned}$$

e.g.
$$\begin{aligned} f(2,k) &= (k-1)\ (k-1+1) = (k-1)\ k \\ f(3,k) &= (k-1)\ ((k-1)^2 + 1) = (k-1)\ (k^2-2k) \\ f(4,k) &= (k-1)((k-1)^3 + 1) = (k-1)\ (k^3 - 3k^2 + 3k) \end{aligned}$$

✓