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New + final topic for the class: **Posets** (Stanley Ch. 3, Ardila §4)

Def'n A partially ordered set or poset $(P, \leq)$ is a binary relation $x \leq y$ on a set P which is

- reflexive $x \leq x$
- antisymmetric $x \leq y,\ y \leq x \Rightarrow x = y$
- transitive $x \leq y,\ y \leq z \Rightarrow x \leq z$

Examples

① $([n], \leq)$ $\longleftarrow 1\ 2\ \cdots\ n \longrightarrow$

$\cap$

$(\mathbb{N}, \leq)$ $\longleftarrow 0\ 1\ 2\ 3 \cdots \longrightarrow$ ← totally (or "linearly") order sets

$\cap$

$(\mathbb{Z}, \leq)$ $\longleftarrow \cdots -2\ -1\ 0\ 1\ 2 \cdots \longrightarrow$ (also called chains)

$\cap$

$(\mathbb{Q}, \leq)$ $\longleftarrow \mathbb{Q} \longrightarrow$

$\cap$

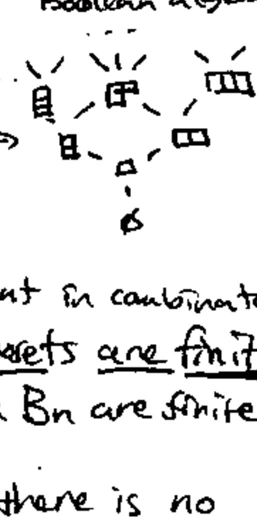
$(\mathbb{R}, \leq)$ $\longleftarrow \mathbb{R} \longrightarrow$

② For a set S, $(2^S, \subseteq)$ = Boolean algebra on {all subsets of S} (also 'Boolean lattice') w/ $X \leq Y$ if $X \subseteq Y$

When $S = [n]$, we write $B_n := 2^{[n]}$ ("nth finite Boolean algebra")

e.g. $B_1 = \begin{matrix}\{1\}\\ | \\ \emptyset\end{matrix}$, $B_2 = \begin{matrix} & \{1,2\} & \\ \{1\} & & \{2\} \\ & \emptyset & \end{matrix}$, etc...

③ $\mathbb{Y}$ = Young's lattice of all partitions, ordered by containment of Young diagrams ⟶



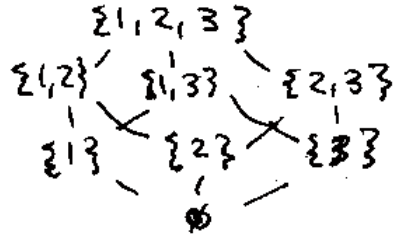
Although some infinite posets are very significant in combinatorics (like $\mathbb{Y}$), to simplify things we will assume all posets are finite from now on! (In examples above, $[n]$ and B_n are finite.)

We write $x \lessdot y \in P$ to mean that $x < y$ and there is no $z \in P$ with $x < z < y$. We say y covers x in this case. If P is finite, then $\leq$ is the reflexive, transitive closure of the cover relation $\lessdot$.

This means we can represent a poset P by its Hasse diagram:

$P =$  ← edges of Hasse diagram = cover relations ↗  means $x \lessdot y$

So e.g. we draw Boolean algebra B_3 as:



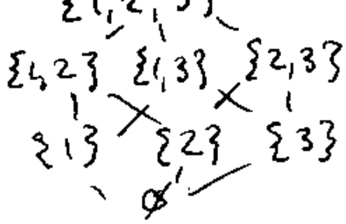
$\{2\} \subseteq \{1,2,3\}$, but don't draw edge since not a cover relation

Why are we interested in posets? Many reasons!
One: we can get sequences of numbers from posets.

DEF'N Let P be a poset. A chain $C \subseteq P$ is a totally ordered subset of P. It is maximal iff maximal by inclusion. We say P is graded if we can write $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_n$ so that every maximal chain has form $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$, where $x_i \in P_i$. In this case, $\exists$ unique rank function $\rho: P \to \{0,1,\dots,n\}$ satisfying $\rho(x) = 0$ iff x is minimal in P,
and $\rho(y) = \rho(x)+1$ if $x \lessdot y \in P$. ← also will write 'rank(p)'

Define rank generating fn $F(P,x) = \sum_{p \in P} x^{\rho(p)}$

Examples ① $F(B_n, x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$



$\cdots P_3 \quad \binom{3}{3} = 1 \cdot x^3$

$\cdots P_2 \quad \binom{3}{2} = 3 \cdot x^2$

$\cdots P_1 \quad \binom{3}{1} = 3 \cdot x$ ✓

$\cdots P_0 \quad \binom{3}{0} = 1$

Boolean algebra graded by cardinality

(2) Let $\Pi_n = \{$set partitions of $[n]\}$. Define po $\leq$ on Π_n by $\pi \leq \pi' \in \Pi_n$ iff π refines π', i.e., with $\pi = \{S_1, S_2, \ldots, S_k\}$ and $\pi' = \{S'_1, \ldots, S'_\ell\}$ every S'_j is a union of some of the S_i.

e.g. $\Pi_3 =$

123	P_{n-1}	$S(3,1) = 1 \cdot x^2$
1|23 2|13 3|12	P_1	$S(3,2) = 3 \cdot x$
1|2|3	P_0	$S(3,3) = 1$

Stirling #'s of 2nd kind

Π_n is graded with $\operatorname{rank}(\pi) = n - \#\text{blocks}(\pi)$

So $F(\Pi_n, x) = \sum_{k=0}^{n-1} S(n, n-k)\, x^k$ ✓

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(3) There are several interesting partial orders on sym. gp. S_n. Let $T = \{(i,j) : 1 \leq i < j \leq n\} \subseteq S_n$ be transpositions in S_n.

Define $\ell_T(w) :=$ minimal length of an expression for w as a product of elements of T.

e.g. $\ell_T((1,3,2)) = 2$ since $(1,3,2) = (1,2)\cdot(1,3)$.

Define absolute order $\leq_{abs}$ on S_n by cover relations:

$$w \lessdot u \iff u = wt \text{ for some } t \in T + \ell_T(u) = \ell_T(w) + 1$$

e.g.

(132) (123)	P_2	$c(3,1) = 2 \cdot x^2$
(12) (13) (23)	P_1	$c(3,2) = 3 \cdot x$
$e = (1)(2)(3)$	P_0	$c(3,3) = 1$

Stirling #'s of 1st kind

NOTE: $\ell_T(w) = \operatorname{rank}(w) = n - \#\text{cycles}(w)$

So $P = (S_n, \leq_{abs})$ is graded w/ $F(P, x) = \sum_{k=0}^{n-1} c(n, n-k)\, x^k = (x+1)(2x+1)\cdots((n-1)x+1)$.

Rmk Let $S = \{(i, i+1) : 1 \leq i < n\} \subseteq S_n$ be set of simple transpositions

Can define weak order $\leq_{weak}$ analogously, w/ $\ell_S(w) =$ min. length of product of S's that $= w$

321

231 312

213 132

123

Then $P = (S_n, \leq_{weak})$ is graded w/ $\operatorname{rank}(w) = \ell_S(w) = \operatorname{inv}(w)$

So $F(P, q) = \sum_{w \in S_n} q^{\operatorname{inv}(w)} = [n]_q!$.

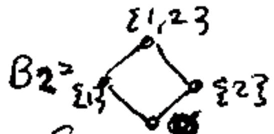
Lattices: an important class of posets

Def'n Say P is a meet semilattice if every $x, y \in P$ have some element $x \wedge y \in P$, their meet, which is a greatest lower bound: for all $z \in P$, if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y \leq x, y$. Dually, it is a join semilattice if $\forall x, y \in P$, $\exists$ a join $x \vee y \in P$ which is a least upper bound: $\forall z \in P$ with $z \geq x, y$ have $z \geq x \vee y \geq x, y$. It is a lattice if it is both a join and meet semilattice.

Examples ① Finite chains $[n]$ are graded lattices.

② Finite Boolean lattices B_n are graded lattices

with $S \wedge T = S \cap T$
$S \vee T = S \cup T$



← Most important example for intuition!

③ The pentagon lattice $P =$  is a lattice, but not graded.

very useful!

④ Prop. A finite meet semilattice $(P, \leq)$ always has a $\hat{0}$ (= unique minimum element), and if it has a $\hat{1}$ (= maximum elt.) then it is a lattice.

Proof: Check that $(\cdots((x_1 \wedge x_2) \wedge x_3) \cdots \wedge x_\ell)$ is a greatest lower bound for any (finite, non-empty) subset $S = \{x_1, x_2, \ldots, x_\ell\}$ in a meet semilattice. Hence if $P = \{p_1, \ldots, p_\ell\}$, then $\hat{0} = p_1 \wedge \cdots \wedge p_\ell$ exists in P.

Also, if P has a $\hat{1}$, then given $x, y \in P$ the set $\{x_1, \ldots, x_\ell\}$ of all upper bounds for x, y (i.e., $x_i \geq x, y$) is nonempty (since it contains $\hat{1}$), and one can check that we can then define $x \vee y := x_1 \wedge \cdots \wedge x_\ell$. ▨

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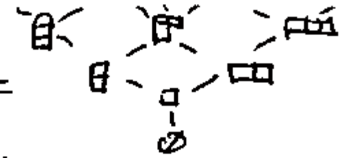
(5) Young's lattice of partitions is an infinite, graded, lattice $\mathbb{Y} =$



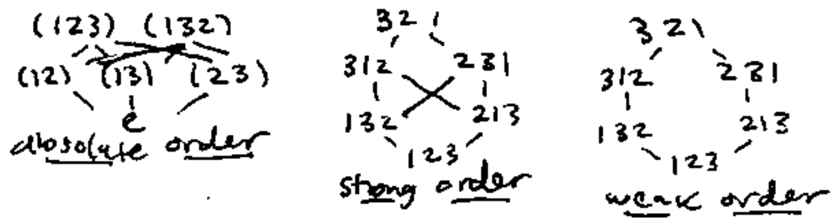
w/ $\lambda \wedge \mu = \lambda \cap \mu$, $\lambda \vee \mu = \lambda \cup \mu$, $\mathrm{rank}(\lambda) = |\lambda|$, so $F(\mathbb{Y}, x) = \sum_{n \geq 0} p(n) x^n$

(6) The (set) partition lattice $\Pi_n = \{$set partitions of $[n]\}$ is a graded lattice with $\pi \wedge \pi' =$ common refinement of π, π'

$\pi \vee \pi' =$ transitive closure of blocks of π, π'

(7) We defined absolute order and weak order on sym. gp S_n. There is a 3rd order on S_n, strong (Bruhat) order, a kind of hybrid of absolute + weak order which I won't even define:

e.g. $n=3$

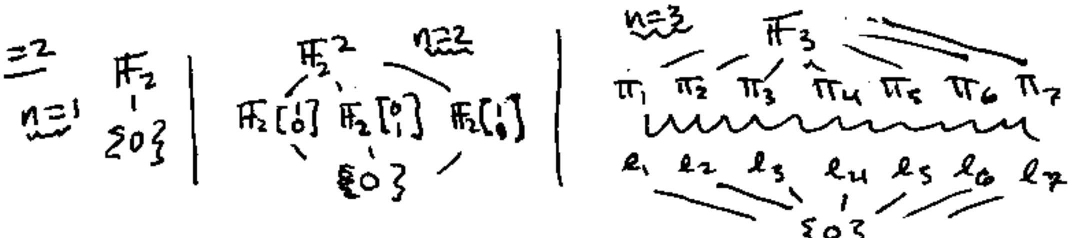


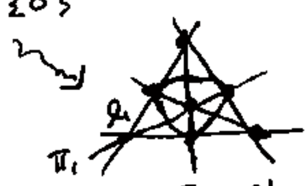
Absolute order + strong order are not lattices (check) but weak order is a lattice (not obvious result!).

(8) $B_n(q) = \mathcal{L}_n(q) = \mathcal{L}(\mathbb{F}_q^n) = \{$all $\mathbb{F}_q$-linear subspaces $V \subseteq \mathbb{F}_q^n\}$
$=$ (finite) vector space lattice

ordered by $\subseteq$ (containment) are graded lattices
with $U \wedge W = U \cap W$
and $U \vee W = U + W$ $(= \{u+w : u \in U, w \in W\})$
and $\mathrm{rank}(U) = \dim_{\mathbb{F}_q}(U)$

e.g. $q=2$





"Fano plane"

Note that rank generating fn is

$$F(B_n(q), x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$$

(9) There are several important <u>operations</u> on posets P and Q:

- <u>disjoint union</u> $P \uplus Q$ = poset on $P \cup Q$ w/ $p \in P$, $q \in Q$ <u>incomparable</u>
- (<u>Cartesian</u>) <u>product</u> $P \times Q$ w/ <u>component-wise</u> <u>order</u> $(p_1, q_1) \le (p_2, q_2)$ $\Leftrightarrow p_1 \le p_2$ and $q_1 \le q_2$
- <u>dual</u> <u>poset</u> P^* = same elt's as P but $\le$ upside-down

<u>e.g.</u> $P =$  $Q =$  $\Rightarrow P \uplus Q =$ , $P \times Q =$ , $P^* =$ 

<u>Prop.</u> P, Q lattices $\Rightarrow P \times Q$ lattice, P^* lattice

graded $\Rightarrow$ graded, w/ $F(P \times Q, x) = F(P, x) \cdot F(Q, x)$.

(10) <u>Def'n</u> An <u>order ideal</u> $I \subseteq P$ of a poset P is a subset closed under <u>going</u> <u>down</u>: i.e., $p \in I$ and $p' \le p \Rightarrow p' \in I$.

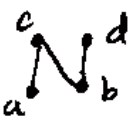
$J(P) := \{$ the <u>lattice</u> of all order ideals $I \subseteq P$ $\}$ ordered via $\subseteq$ with

$$I_1 \wedge I_2 = I_1 \cap I_2$$
$$I_1 \vee I_2 = I_1 \cup I_2$$

is a <u>graded</u> <u>lattice</u>.

and rank $(I) = \# I \rightsquigarrow F(J(P), x) = \sum_{I \text{ order ideal}} x^{\# I}$

In fact it is a <u>distributive</u> <u>lattice</u>, i.e., $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(because $\cap$ and $\cup$ satisfy these)

<u>e.g.</u> $P =$ 

$\Rightarrow J(P) =$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & abcd \arrow[dl, dash] \arrow[dr, dash] & & & \\
abc \arrow[dr, dash] & & abd \arrow[dl, dash] \arrow[dr, dash] & & \\
 & ab \arrow[dl, dash] \arrow[dr, dash] & & bd \arrow[dl, dash] & \\
a \arrow[dr, dash] & & b \arrow[dl, dash] & & \\
 & \emptyset & & &
\end{tikzcd}$$

<u>Prop:</u> $J(P \uplus Q) = J(P) \times J(Q)$.

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Distributive lattices (Stanley §3.4)

Prop. In any lattice L,

(a) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z) \quad \forall x, y, z \in L$

(b) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in L$

and equality holds in (a) $\forall x,y,z \Leftrightarrow$ equality holds for (b) $\forall x, y, z$.

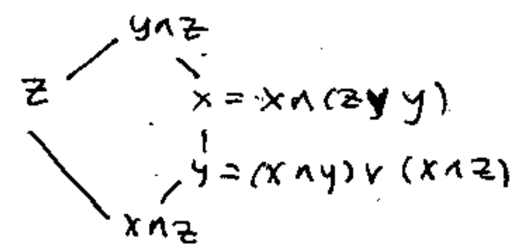
Pf. Skipped. Exercise for you.— ☒

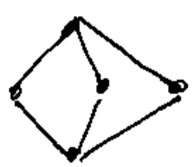
Def'n L is a distributive lattice if equality holds for (a) + (b) in the previous prop. $\forall x, y, z \in L$.

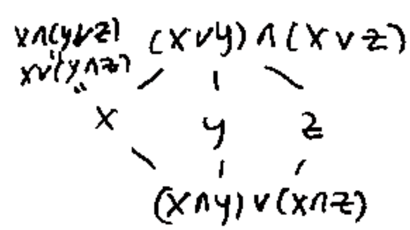
Examples ① P a poset, $J(P) = \{$order ideals $I \subseteq J(P)\}$ ordered by containment is a distr. lattice.

② L_1, L_2 distributive $\Rightarrow L_1 \times L_2$ distributive.

③  is not distributive:



④  is not distr.:



⑤ Young's lattice $\mathbb{Y}$ of all partitions ordered by containment is an infinite distr. lattice.

Rmk: Birkhoff showed that a lattice L is distr. $\Leftrightarrow L$ has no sublattice isomorphic to ③  or ④ .

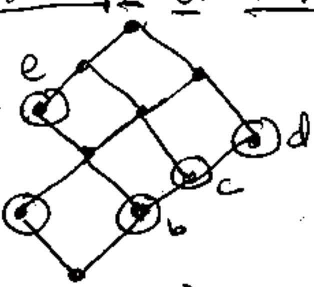
More importantly for us, Birkhoff proved the following:

Thm (Fundamental Thm. of Finite Distributive Lattices)
Every finite distributive lattice L is isomorphic to $J(P)$ for a poset P defined uniquely (up to isomorphism) namely $P \cong \mathrm{Irr}(L) := \{\text{the join irreducible } p \in L\}$.

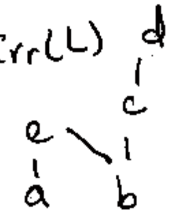
w/ the induced partial order as a subposet of L

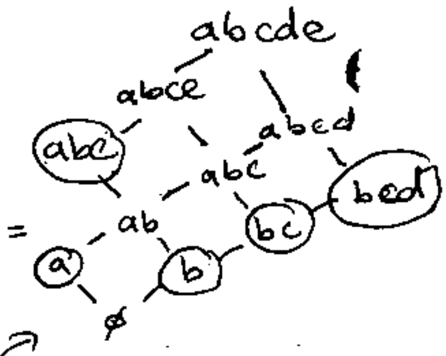
say p is join irreducible if $p = x_1 \vee \cdots \vee x_\ell$ for some $\{x_1, \ldots, x_\ell\} \subseteq L$
$\Downarrow$
$p = x_i$ for some i.

Example of FTFDL:

$L =$ 

is distributive, w/ elements of $P = \mathrm{Irr}(L)$ labeled. (note: p is join irreducible $\Leftrightarrow$ p covers exactly one element in L)

$P = \mathrm{Irr}(L) =$ 

$J(P) =$ 

NOTE: That the join irreducibles in $J(P) = \{\text{principal order ideals}\}$

i.e. $I = \{q : q \le p\}$ for some $p \in P$

principal order ideal

non-principal order ideal

Pf of Birkhoff's FTFDL:

Given L, finite distributive lattice, define maps

$$L \underset{g}{\overset{f}{\rightleftarrows}} J(P) \text{ where } P = Irr(L)$$

$$x \longmapsto f(x) := \{p \in Irr(L) : p \leq x\}$$

$$g(I) := p_1 \vee \cdots \vee p_\ell \longleftarrow\!\shortmid \; I = \{p_1, \ldots, p_\ell\}$$

It's not hard to see both f, g order-preserving: i.e., $x \leq y \Rightarrow f(x) \subseteq f(y)$, $I \subseteq I' \Rightarrow g(I) \leq g(I')$

We claim that in any finite lattice (not nec. distributive) one has $g(f(x)) = \bigvee_{\substack{p \in Irr(L) \\ p \leq x}} p = x$

Certainly $\bigvee_{\substack{p \in Irr(L) \\ p \leq x}} p \leq x$ since each $p \leq x$, but also one can write $x = p_1 \vee p_2 \vee \cdots \vee p_\ell$ with each p_i join irreducible, using downwards induction on $x \in L$ (either $x \in Irr(L)$, or write $x = x_1 \vee x_2$ with $x_1 < x$, $x_2 < x$, and repeat)

Hence indeed $x = \bigvee_{\substack{p \in Irr(L) \\ p \leq x}} p = g(f(x))$.

On the other hand, $f(g(\underbrace{I}_{\{p_1, \ldots, p_\ell\}})) = \{q \in Irr(L) : q \leq p_1 \vee \cdots \vee p_\ell\} \supseteq I$.

But, in a distr. lattice, $q \leq p_1 \vee \cdots \vee p_\ell \Rightarrow q = q \wedge (p_1 \vee \cdots \vee p_\ell)$

using distributivity $\rightsquigarrow$ $= (q \wedge p_1) \vee \cdots \vee (q \wedge p_\ell)$

since $q \in Irr(L)$ $\rightsquigarrow$ $\Rightarrow q = q \wedge p_i$ for some i

if $x = x \wedge y$ then $x \leq y$ $\longrightarrow$ $\Rightarrow q \leq p_i \in I$

since I is an order ideal $\rightsquigarrow$ $\Rightarrow q \in I$.

Hence, $f(g(I)) = \{q \in Irr(L) : q \leq p_1 \vee \cdots \vee p_\ell\} \subseteq I$, and so $f(g(I)) = I$.

These f and g give isomorphisms between L and $J(P)$. ∎