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New + final topic for the class: **Posets** (Stanley Ch. 3, Ardila §4)

Def'n A partially ordered set or poset $(P, \leq)$ is a binary relation $x \leq y$ on a set P which is

- reflexive $x \leq x$
- antisymmetric $x \leq y,\ y \leq x \Rightarrow x = y$
- transitive $x \leq y,\ y \leq z \Rightarrow x \leq z$

Examples

① $([n], \leq)$ $\longleftarrow 1\ 2\ \cdots\ n \longrightarrow$

$\cap$

$(\mathbb{N}, \leq)$ $\longleftarrow 0\ 1\ 2\ 3 \cdots \longrightarrow$ ← totally (or "linearly") order sets

$\cap$

$(\mathbb{Z}, \leq)$ $\longleftarrow \cdots -2\ -1\ 0\ 1\ 2 \cdots \longrightarrow$ (also called chains)

$\cap$

$(\mathbb{Q}, \leq)$ $\longleftarrow \mathbb{Q} \longrightarrow$

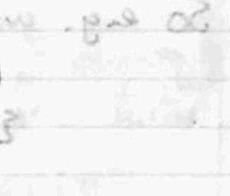
$\cap$

$(\mathbb{R}, \leq)$ $\longleftarrow \mathbb{R} \longrightarrow$

② For a set S, $(2^S, \subseteq)$ = Boolean algebra on {all subsets of S} (also 'Boolean lattice') w/ $X \leq Y$ if $X \subseteq Y$

when $S = [n]$, we write $B_n := 2^{[n]}$ ("n^{th} finite Boolean algebra")

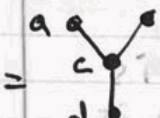
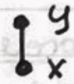
e.g. $B_1 = \begin{matrix} \{1\} \\ | \\ \phi \end{matrix}$, $B_2 = \begin{matrix} & \{1,2\} & \\ \{1\} & & \{2\} \\ & \phi & \end{matrix}$, etc...

③ $\mathbb{Y}$ = Young's lattice of all partitions, ordered by containment of Young diagrams $\longrightarrow$ 

Although some infinite posets are very significant in combinatorics (like $\mathbb{Y}$), to simplify things we will assume all posets are finite from now on! (In examples above, $[n]$ and B_n are finite.)

We write $x \lessdot y \in P$ to mean that $x < y$ and there is no $z \in P$ with $x < z < y$. We say y covers x in this case. If P is finite, then $\leq$ is the reflexive, transitive closure of the cover relation $\lessdot$.

This means we can represent a poset P by its <u>Hasse diagram</u>:

$P =$  ← edges of Hasse diagram = cover relations ↗  means $x \lessdot y$

So e.g. we draw [redacted] Boolean algebra B_3 as:

$\{1,2,3\}$

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$

$\{1\}$ $\{2\}$ $\{3\}$

$\emptyset$

$\{2\} \subseteq \{1,2,3\}$, but <u>don't</u> draw edge since <u>not</u> a <u>cover relation</u>

Why are we interested in posets? <u>Many reasons</u>!
One: we can get sequences of numbers from posets.

<u>DEF'N</u> Let P be a poset. A <u>chain</u> $C \subseteq P$ is a totally ordered subset of P. It is <u>maximal</u> iff maximal by inclusion.
We say P is <u>graded</u> if we can write $P = P_0 \sqcup P_1 \sqcup \cdots \sqcup P_n$ so that every maximal chain has form $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n$, where $x_i \in P_i$. In this case, $\exists$ unique <u>rank function</u> $\rho: P \to \{0,1,\dots,n\}$ satisfying $\rho(x) = 0$ iff x is <u>minimal</u> in P,
and $\rho(y) = \rho(x)+1$ if $x \lessdot y \in P$.

Define <u>rank generating fn</u> $F(P,x) = \sum_{p\in P} x^{\rho(p)}$ ↙ also will write 'rank(p)'

<u>Examples</u> ① $F(B_n, x) = \sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$

$\{1,2,3\}$ --- P_3 $\binom{3}{3} = 1 \cdot x^3$

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$ --- P_2 $\binom{3}{2} = 3 \cdot x^2$

$\{1\}$ $\{2\}$ $\{3\}$ --- P_1 $\binom{3}{1} = 3 \cdot x$ ✓ Boolean algebra <u>graded</u> by cardinality

$\emptyset$ --- P_0 $\binom{3}{0} = 1$

(2) Let $\Pi_n = \{\text{set partitions of } [n]\}$. Define p.o. $\le$ on Π_n by $\pi \le \pi' \in \Pi_n$ iff π <u>refines</u> π', i.e., with $\pi = \{S_1, S_2, \dots, S_k\}$ and $\pi' = \{S'_1, \dots, S'_\ell\}$ every S'_j is a union of some of the S_i.

e.g. $\Pi_3 =$

	123		P_{n-1}: $S(3,1) = 1 \cdot x^2$
1|23	2|13	3|12	P_1: $S(3,2) = 3 \cdot x$
	1|2|3		P_0: $S(3,3) = 1$

Stirling #'s of 2nd kind

Π_n is graded with $\operatorname{rank}(\pi) = n - \#\,\text{blocks}(\pi)$

So $F(\Pi_n, x) = \sum_{k=0}^{n-1} S(n, n-k)\, x^k$ ✓

11/15 (3) There are several interesting partial orders on sym. gp. S_n.

Let $T = \{(i,j) : 1 \le i < j \le n\} \subseteq S_n$ be <u>transpositions</u> in S_n

Define $\ell_T(w) :=$ minimal length of an expression for w as a product of elements of T.

e.g. $\ell_T((1,3,2)) = 2$ since $(1,3,2) = (1,2)\cdot(1,3)$.

Define <u>absolute order</u> $\le_{abs}$ on S_n by <u>cover relations</u>:

$w \lessdot u \iff u = wt$ for some $t \in T$ + $\ell_T(u) = \ell_T(w) + 1$

e.g.

(132)	(123)		P_2: $c(3,1) = 2 \cdot x^2$
(12)	(13)	(23)	P_1: $c(3,2) = 3 \cdot x$
	$e = (1)(2)(3)$		P_0: $c(3,3) = 1$

Stirling #'s of 1st kind

NOTE: $\ell_T(w) = \operatorname{rank}(w) = n - \#\,\text{cycles}(w)$

So $P = (S_n, \le_{abs})$ is graded w/ $F(P, x) = \sum_{k=0}^{n-1} c(n, n-k)\, x^k = (x+1)(2x+1)\cdots((n-1)x+1)$.

Rmk Let $S = \{(i, i+1) : 1 \le i < n\} \subseteq S_n$ be set of <u>simple transpositions</u>

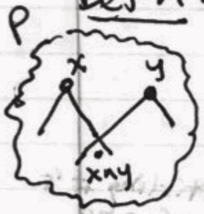
Can define <u>weak order</u> $\le_{weak}$ analogously, w/ $\ell_S(w) =$ min. length of product of S's that $= w$

	321	
231		312
213		132
	123	

Then $P = (S_n, \le_{weak})$ is graded w/ $\operatorname{rank}(w) = \ell_S(w) = \operatorname{inv}(w)$

So $F(P, q) = \sum_{w \in S_n} q^{\operatorname{inv}(w)} = [n]_q!$.

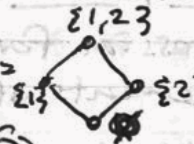
Lattices: an important class of posets

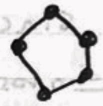


Def'n Say P is a *meet semilattice* if every $x, y \in P$ have some element $x \wedge y \in P$, their *meet*, which is a *greatest lower bound*: for all $z \in P$, if $z \le x$ and $z \le y$ then $z \le x \wedge y \le x, y$. Dually, it is a *join semilattice* if $\forall x, y \in P$, $\exists$ a *join* $x \vee y \in P$ which is a *least upper bound*: $\forall z \in P$ with $z \ge x, y$ have $z \ge x \vee y \ge x, y$. It is a *lattice* if it is both a join and meet semilattice.

Examples ① Finite chains $[n]$ are *graded* lattices.

② Finite Boolean lattices B_n are *graded lattices* with $S \wedge T = S \cap T$, $S \vee T = S \cup T$ ← Most important example for intuition!

$B_2 =$ 

③ The *pentagon lattice* $P =$  is a lattice, but *not* graded.

very useful! ↘ ④ *Prop.* A finite meet semilattice $(P, \le)$ always has a $\hat{0}$ (= unique *minimum* element), and if it has a $\hat{1}$ (= *maximum* elt.) then it is a lattice.

Proof: Check that $(\cdots((x_1 \wedge x_2) \wedge x_3) \cdots \wedge x_\ell)$ is a *greatest lower bound* for any finite (non-empty) subset $S = \{x_1, x_2, \ldots, x_\ell\}$ in a meet semilattice. Hence if $P = \{p_1, \ldots, p_\ell\}$, then $\hat{0} = p_1 \wedge \cdots \wedge p_\ell$ exists in P.

Also, if P has a $\hat{1}$, then given $x, y \in P$ the set $\{x_1, \ldots, x_\ell\}$ of all upper bounds for x, y (i.e., $x_i \ge x, y$) is *nonempty* (since it contains $\hat{1}$), and one can check that we can then define $x \vee y := x_1 \wedge \cdots \wedge x_\ell$. ▨

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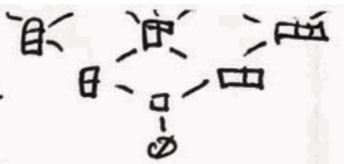
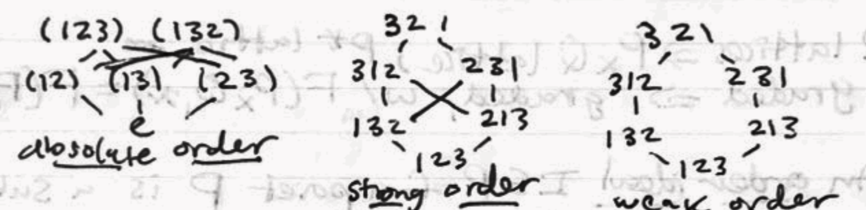
(5) Young's lattice of partitions $\mathbb{Y}$ = 
is an infinite, graded, lattice
w/ $\lambda \wedge \mu = \lambda \cap \mu$, $\lambda \vee \mu = \lambda \cup \mu$, $\mathrm{rank}(\lambda) = |\lambda|$, so $F(\mathbb{Y}, x) = \sum_{n \geq 0} p(n) x^n$

(6) The (set) partition lattice $\Pi_n = \{$set partitions of $[n]\}$
is a graded lattice with $\pi \wedge \pi' =$ common refinement of π, π'
$\pi \vee \pi' =$ transitive closure of blocks of π, π'

(7) We defined absolute order and weak order on sym. gp S_n.
There is a 3^{rd} order on S_n, Strong (Bruhat) order, a kind of hybrid of absolute + weak order which I won't even define:

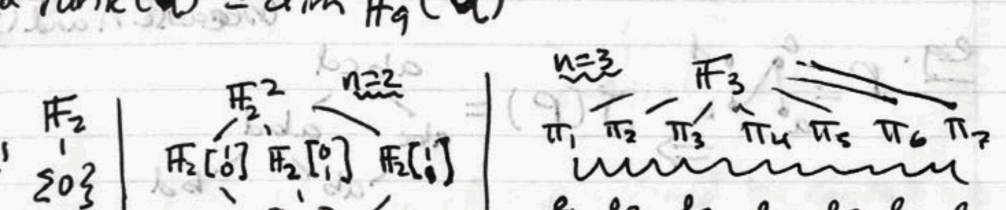
e.g. $n=3$



Absolute order + strong order are not lattices (check)
but weak order is a lattice (not obvious result!).

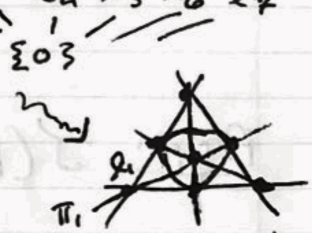
(8) $B_n(q) = \mathcal{L}_n(q) = \mathcal{L}(\mathbb{F}_q^n) = \{$all $\mathbb{F}_q$-linear subspaces $V \subseteq \mathbb{F}_q^n\}$
$=$ (finite) vector space lattice
ordered by $\subseteq$ (containment) are graded lattices
with $U \wedge W = U \cap W$
and $U \vee W = U + W$ $(= \{u+w : u \in U, w \in W\})$
and $\mathrm{rank}(U) = \dim_{\mathbb{F}_q}(U)$

e.g. $q=2$



Note that rank generating fn is

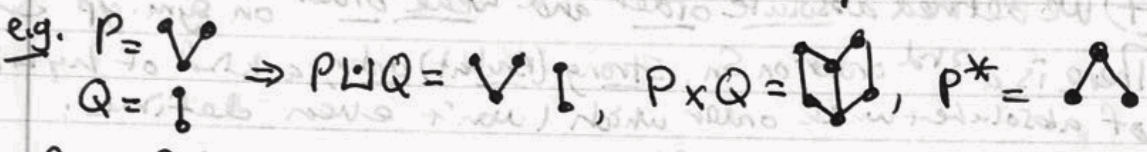
$$F(B_n(q), x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k.$$



"Fano plane"

(9) There are several important operations on posets P and Q:

- disjoint union $P \sqcup Q$ = poset on $P \cup Q$ w/ $p \in P, q \in Q$ incomparable
- (Cartesian) product $P \times Q$ w/ component-wise order $(p_1, q_1) \le (p_2, q_2) \iff p_1 \le p_2$ and $q_1 \le q_2$
- dual poset P^* = same elt's as P but $\le$ upside-down

e.g. 

Prop. P, Q lattices $\Rightarrow P \times Q$ lattice, P^* lattice
graded $\Rightarrow$ graded, w/ $F(P \times Q, x) = F(P, x) \cdot F(Q, x)$.

P (10) Def'n An order ideal $I \subseteq P$ of a poset P is a subset closed under going down: i.e., $p \in I$ and $p' \le p \Rightarrow p' \in I$.

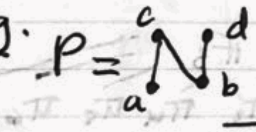
$J(P) := \{$ the lattice of all order ideals $I \subseteq P$ ordered via $\subseteq \}$ with

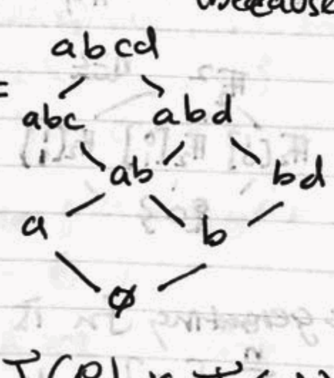
$$I_1 \wedge I_2 = I_1 \cap I_2$$
$$I_1 \vee I_2 = I_1 \cup I_2$$

is a graded lattice.

and $\mathrm{rank}(I) = \# I \rightsquigarrow F(J(P), x) = \sum_{I \text{ order ideal}} x^{\# I}$

In fact it is a distributive lattice, i.e., $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(because $\cap$ and $\cup$ satisfy these)

e.g. $P =$ 

$\Rightarrow J(P) =$ 

Prop: $J(P \sqcup Q) = J(P) \times J(Q)$.

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Distributive lattices (Stanley §3.4)

Prop. In any lattice L,

(a) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z) \quad \forall x, y, z \in L$

(b) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in L$

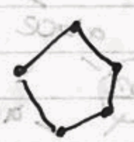
and equality holds in (a) $\forall x,y,z \Leftrightarrow$ equality holds for (b) $\forall x,y,z$.

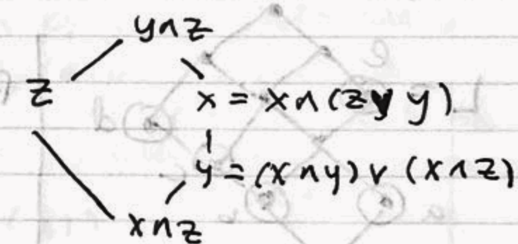
Pf: Skipped. Exercise for you... ☒

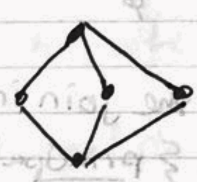
Def'n L is a distributive lattice if equality holds for (a) + (b) in the previous prop. $\forall x,y,z \in L$.

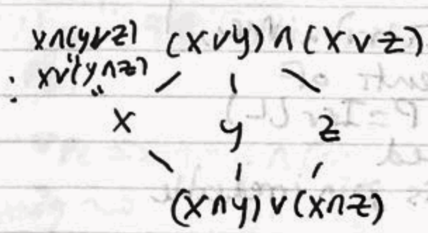
Examples ① P a poset, $J(P) = \{$order ideals $I \subseteq J(P)\}$ ordered by containment is a distr. lattice.

② L_1, L_2 distributive $\Rightarrow L_1 \times L_2$ distributive.

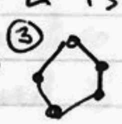
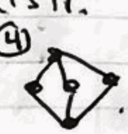
③  is not distributive:



④  is not distr.:



⑤ Young's lattice $\mathbb{Y}$ of all partitions ordered by containment is an infinite distr. lattice.

Rmk: Birkhoff showed that a lattice L is distr. $\Leftrightarrow$ L has no sublattice isomorphic to ③  or ④ .

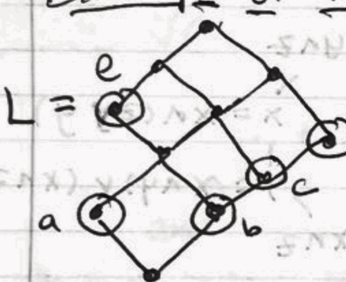
More importantly for us, Birkhoff proved the following:

<u>Thm</u> (Fundamental Thm. of Finite Distributive Lattices)
Every <u>finite</u> <u>distributive</u> <u>lattice</u> L is isomorphic to $J(P)$ for a poset P defined uniquely (up to isomorphism) namely $P \cong \mathrm{Irr}(L) := \{$ the <u>join</u> <u>irreducible</u> $p \in L\}$

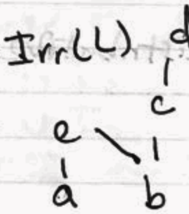
w/ the <u>induced</u> <u>partial</u> <u>order</u> as a subposet of L

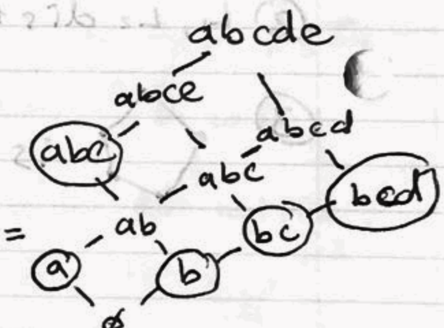
say p is join <u>irreducible</u> if $p = x_1 \vee \cdots \vee x_\ell$ for some $\{x_1, \ldots, x_\ell\} \subseteq L$
$\Downarrow$
$p = x_i$ for some i.

<u>Example</u> <u>of</u> <u>FTFDL</u>:

$L =$ 

is distributive, w/ elements of $P = \mathrm{Irr}(L)$ labelled.
(<u>note</u>: p is join irreducible $\Leftrightarrow$ p covers exactly one element in L)

$P = \mathrm{Irr}(L) =$ 

$J(P) =$ 

<u>NOTE</u>: That the join irreducibles in $J(P) = \{$ <u>principal</u> <u>order</u> <u>ideals</u> $\}$
i.e. $I = \{q : q \le p\}$ for some $p \in P$

principal order ideal


non-principal order ideal

Pf of Birkhoff's FTFDL:

Given L, finite distributive lattice, define maps

$$L \underset{g}{\overset{f}{\rightleftarrows}} J(P) \text{ where } P = Irr(L)$$

$$x \longmapsto f(x) := \{p \in Irr(L) : p \leq x\}$$

$$g(I) := p_1 \vee \cdots \vee p_\ell \longleftarrow\!\!\shortmid\; I = \{p_1, \ldots, p_\ell\}$$

It's not hard to see both f, g order-preserving: i.e., $I \subseteq I' \Rightarrow g(I) \leq g(I')$, $x \leq y \Rightarrow f(x) \subseteq f(y)$

We claim that in any finite lattice (not nec. distributive)

one has $g(f(x)) = \bigvee_{\substack{p \in Irr(L) \\ p \leq x}} p = x$

Certainly $\bigvee_{\substack{p \in Irr(L) \\ p \leq x}} p \leq x$ since each $p \leq x$, but also one can write $x = p_1 \vee p_2 \vee \cdots \vee p_\ell$ with each p_i join irreducible, using downwards induction on $x \in L$ (either $x \in Irr(L)$, or write $x = x_1 \vee x_2$ with $x_1 < x$, $x_2 < x$, and repeat)

Hence indeed $x = \bigvee_{\substack{p \in Irr(L) \\ p \leq x}} p = g(f(x))$.

On the other hand, $f(g(\underset{\{p_1, \ldots, p_\ell\}}{I})) = \{q \in Irr(L) : q \leq p_1 \vee \cdots \vee p_\ell\} \supseteq I$.

But, in a distr. lattice, $q \leq p_1 \vee \cdots \vee p_\ell \Rightarrow q = q \wedge (p_1 \vee \cdots \vee p_\ell)$

using distributivity $\rightsquigarrow$ $= (q \wedge p_1) \vee \cdots \vee (q \wedge p_\ell)$

since $q \in Irr(L)$ $\rightsquigarrow$ $\Rightarrow q = q \wedge p_i$ for some i

if $x = x \wedge y$ then $x \leq y$ $\longrightarrow$ $\Rightarrow q \leq p_i \in I$

since I is an order ideal $\rightsquigarrow$ $\Rightarrow q \in I$.

Hence, $f(g(I)) = \{q \in Irr(L) : q \leq p_1 \vee \cdots \vee p_\ell\} \subseteq I$, and so $f(g(I)) = I$.

These f and g give isomorphisms between L and $J(P)$. ▨