IV.22.

Möbius functions and Möbius inversion (Stanley §3.6, 3.7)

Let's reinterpret inclusion-exclusion as being about the poset $P = B_n = 2^{[n]}$ and functions $f = f_=: P \to R$ ← some commuting e.g. $\mathbb{C}$ where we were given a new function

$$g = f_{\leq} : P \to R \text{ such that } g(S) = \sum_{T \subseteq S} f(T)$$

i.e., $g(y) = \sum_{x \in P} \zeta(x,y) f(x)$, where $\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \text{ in } P \\ 0 & \text{otherwise} \end{cases}$

and we can invert to get

$$f(S) = f_=(S) = \sum_{T \subseteq S} (-1)^{\#S \setminus T} f_{\leq}(T),$$

i.e., $f(y) = \sum_{x \in P} \mu(x,y) g(x)$ where $\mu(\overset{T}{x}, \overset{S}{y}) = \begin{cases} (-1)^{\#y \setminus x} & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$

This same set-up works for all (finite) posets P, once we find what the $\zeta(x,y), \mu(x,y)$ are and where they live...

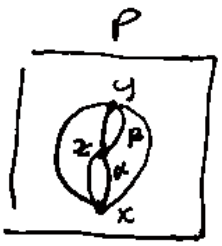
DEF'N The incidence algebra $I(P,R)$ of a (finite) poset P (over a comm. ring R) is the ring of all functions

$$f : \text{Int}(P) \longrightarrow R$$

$$\| \quad \{ \text{(non-empty) intervals } [x,y] := \{z \in P : x \leq z \leq y\} \text{ in } P \}$$

with pointwise addition $(\alpha + \beta)(x,y) = \alpha(x,y) + \beta(x,y)$

and convolution product $(\alpha * \beta)(x,y) = \sum_{z \in [x,y]} \alpha(x,z)\beta(z,y)$.



and identity element $\delta(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq a. \end{cases}$

We'll want to know that the zeta function $\zeta(x,y) := \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$ is invertible in $I(P,R)$.

Prop. $\alpha \in I(P,R)$ has an inverse $\Leftrightarrow \alpha(x,x) \in \overset{\downarrow}{R^\times}$ $\forall x \in P$.

recall: gp. of units, i.e., invertible elts of R

Pf: $\alpha * \beta = \delta \Leftrightarrow \underset{\sum_{z \in [x,y]} \alpha(x,z)\beta(z,y)}{\overset{\shortparallel}{(\alpha*\beta)(x,y)}} = \delta(x,y) = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x \neq y \end{cases} \quad \forall x,y \in P$

which forces $\alpha(x,x)\beta(x,x) = 1$ so $\left\{ \begin{matrix} \alpha(x,x) \in R^\times \\ \text{and } \beta(x,x) = \alpha(x,x)^{-1} \end{matrix} \right\} \forall x \in P$,

and then when $\alpha(x,x) \in R^\times$, the values for $\beta(x,y)$ are uniquely determined by induction on $\#[x,y]$ via the formula

$$\alpha(x,x)\,\beta(x,y) + \sum_{z \in (x,y]} \alpha(x,z)\beta(z,y) = 0 \qquad (x,y] := \{z : x < z \leq y\}$$

$$\Rightarrow \beta(x,y) = -\alpha(x,x)^{-1} \cdot \sum_{z \in (x,y]} \alpha(x,z) \underbrace{\beta(z,y)}_{\#[z,y] < \#[x,y]}$$

Note: we can also get a left-inverse $\beta'(\cdot,\cdot)$ for $\alpha(\cdot,\cdot)$ defined recursively by $\beta'(x,y) = -\alpha(y,y)^{-1} \sum_{z \in [x,y) := \{z : x \leq z < y\}} \beta'(x,z)\,\alpha(z,y)$

but then associativity of $*$ forces

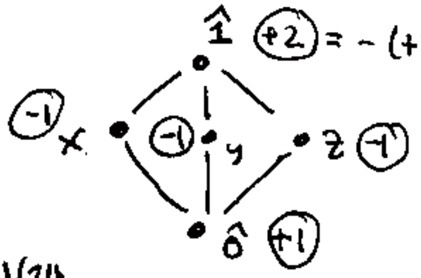
$$\beta' = \beta' * (\alpha * \beta) = (\beta' * \alpha) * \beta = \beta. \checkmark$$ ▨

Cor $\zeta(\cdot,\cdot) \in I(P,R)$ has an inverse, called the Möbius function $\mu = \zeta^{-1}$, defined recursively by $\boxed{\mu(x,x) = 1 \quad \forall x \in P}$

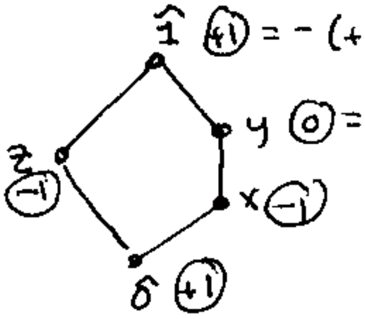
and either $\mu(x,y) = -\sum_{z \in (x,y]} \mu(z,y) \quad \forall x < y$

or $\boxed{\mu(x,y) = -\sum_{z \in [x,y)} \mu(x,z) \quad \forall x < y}$

Examples ① Let's compute $\mu(\hat{0}, p)$ $\forall p$ here (values circled)



$= -(+1-1-1-1)$



$= -(+1-1+0-1)$

$= -(+1-1)$

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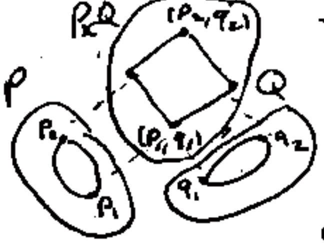
② In a finite chain, $\mu(x,y) = \begin{cases} +1 & \text{if } x=y \\ -1 & \text{if } x \lessdot y \\ 0 & \text{otherwise} \end{cases}$

③ Prop: In a product $P \times Q$, $\mu_{P\times Q}((p_1,q_1),(p_2,q_2)) = \mu_P(p_1,p_2)\mu_Q(q_1,q_2)$

Proof: The function $\alpha(\cdot,\cdot) \in I(P\times Q, R)$ defined by the RHS satisfies the correct initial condition and recurrence: $\alpha((p,q),(p,q)) = \underbrace{\mu_P(p,p)}_{+1}\ \underbrace{\mu_Q(q,q)}_{+1} = +1$ ✓

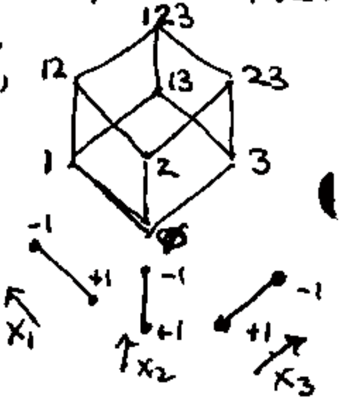


$$\sum_{(p,q)\in[(p_1,q_1),(p_2,q_2)]} \mu_P(p_1,p)\mu_Q(q_1,q) = \Big(\underbrace{\sum_{p\in[p_1,p_2]}}_{=0 \text{ if } p_1<p_2} \mu_P(p,p)\Big)\underbrace{\Big(\sum_{q\in[q_1,q_2]} \mu_Q(q_1,q)\Big)}_{=0 \text{ if } q_1<q_2}$$

$$= 0 \checkmark \text{ if } (p_1,p_2)<(q_1,q_2). \ ▨$$

④ Cor In $B_n = 2^{[n]} \cong [2]^n = [2]\times[2]\times\cdots[2]$,

$\mu(T,S) = (-1)^{\# S\setminus T}$ for $T \subseteq S$



Thm (Möbius inversion formula)

Let P be a poset and $f, g : P \to \underset{\wedge}{R}^{\text{a comm. ring, e.g. } \mathbb{C}}$ related by

$$g(y) = \sum_{x \in P:\, x \leq y} f(x) \quad \forall\, y \in P, \text{ then}$$

$$f(y) = \sum_{x \in P:\, x \leq y} \mu(x,y)\, g(x) \quad \forall\, y \in P.$$

(And *dually*, if we have $g(y) = \sum_{x:\, x \geq y} f(x)$, then $f(y) = \sum_{x:\, x \geq y} \mu(y,x)\, g(x)$.)

Proof: Let $R^P := \{\text{all functions } f : P \to R\}$.

Then $\alpha \in I(P,R)$ *acts* on such an $f \in R^P$ by

$$(f \cdot \alpha)(y) = \sum_{x \in P} f(x)\, \alpha(x,y).$$

Check that $(f \cdot \alpha) \cdot \beta = f \cdot (\alpha * \beta)$ since

$$\begin{aligned}
((f \cdot \alpha) \cdot \beta)(y) &= \sum_{x \in P} (f \cdot \alpha)(x)\, \beta(x,y) \\
&= \sum_{x \in P} \sum_{x' \in P} f(x')\, \alpha(x',x)\, \beta(x,y) \\
&= \sum_{x' \in P} f(x') \Big(\underbrace{\sum_{x \in P} \alpha(x',x)\, \beta(x,y)}_{(\alpha * \beta)(x,y)} \Big) \\
&= (f \cdot (\alpha * \beta))(y) \checkmark
\end{aligned}$$

Then $g(y) = \sum_{x \leq y} f(x) = \sum_{x \in P} f(x)\, \zeta(x,y)$,

i.e., $g = f \cdot \zeta$

$\big\downarrow$ act on right by $\mu = \zeta^{-1}$

$$g \cdot \mu = f, \text{ i.e., } \sum_{x \in P} g(x)\, \mu(x,y) = f(y)$$

$$\| $$

$$\sum_{x \leq y} \mu(x,y)\, g(x). \checkmark \qquad ▨$$

Cor With $P = B_n$, get *Principle of Inclusion-Exclusion*.

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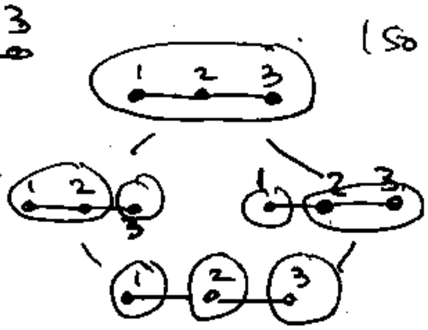
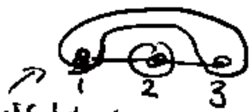
Application of Möbius inversion: Chromatic Polynomials

Defn Let $G=(\overset{[n]}{V},E)$ be a $\overset{\text{connected!}}{}$graph. Say that a $\overset{(set)}{\text{partition}}$ of $[n]$ is G-connected if the restriction of G to each block is connected. Bond lattice Π_G is the sub-poset of the partition lattice Π_n consisting of G-connected partitions, (so ordered by refinement).

Example

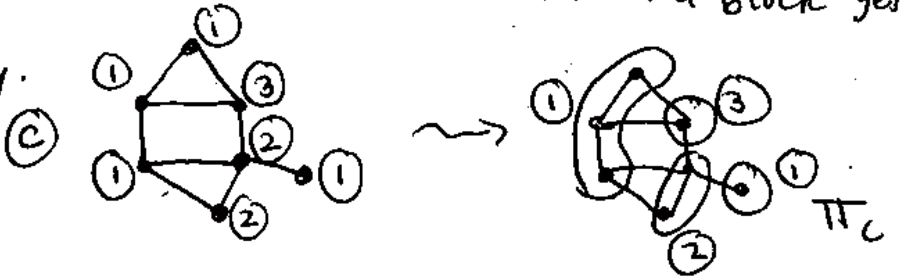
$G =$


$\Pi_G =$


(NOT!


this block is not connected.)

Let $c: V \to \{1,2,3,\ldots\}$ be any coloring of the vertices of G. Associated to c is a G-connected partition Π_c = max. element of Π_G s.t. all vertices in a block get same color.

E.g.


Choose $t \in \mathbb{N}$, the max. # of colors, and let $f,g: \Pi_G \to \mathbb{C}$ be

$$f(\pi) := \#\{\text{colorings } c: V \to \{1,2,\ldots,t\} \text{ s.t. } \Pi_c = \pi\},$$

$$g(\pi) := \#\{\text{colorings } c: V \to \{1,2,\ldots,t\} \text{ s.t. } \Pi_c \geq \pi\}.$$

Observe $g(\pi) = \sum_{\pi' \geq \pi \in \Pi_G} f(\pi')$, but <u>also</u>

$g(\pi) = t^{\#\text{blocks}(\pi)}$, since we can get a coloring c w/ $\pi_c \geq \pi$ by <u>coloring each block independently</u>.

<u>**Cor**</u> For any $\pi \in \Pi_G$, $f(\pi) = \sum_{\pi' \geq \pi \in \Pi_G} \mu(\pi, \pi')\, t^{\#\text{blocks}(\pi')}$,

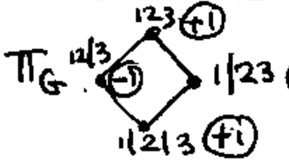
and in particular, w/ $\pi = \hat{0} = \{\{1\}, \{2\}, \dots, \{n\}\}$,

\# of <u>proper</u> colorings $c: V \to \{1, 2, \dots, t\}$ $= \sum_{\pi \in \Pi_G} \mu(\hat{0}, \pi)\, t^{\#\text{blocks}(\pi)}$

no two adjacent vertices get same color

DEFN This is the <u>chromatic polynomial</u> of G, denoted $\chi_G(t)$.

<u>Example</u> $G =$ 

Π_G 

$\leftarrow \mu(\hat{0}, \pi)$ circled

So $\chi_G(t) = +1 \cdot t^3 + (-1-1) \cdot t^2 + 1 \cdot t = t^3 - 2t^2 + t = t(t-1)^2$

<u>**Cor**</u> In the full partition lattice Π_n we have

$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$$

<u>Pf.</u> $\Pi_n = \Pi_{K_n}$ for the <u>complete graph</u> K_n.

But choosing colors 1 at a time, we see that

$$\chi_{K_n}(t) = t(t-1)(t-2) \cdots (t-(n-1)).$$

So $\sum_{\pi \in \Pi_n} \mu(\hat{0}, \pi)\, t^{\#\text{blocks}(\pi)} = t(t-1) \cdots (t-(n-1))$

extract coeff. of $t \rightsquigarrow \mu(\hat{0}, \underset{\{[n]\}}{\overset{\shortparallel}{\hat{1}}}) = (-1) \cdot (-2) \cdots (-(n-1)) = (-1)^{n-1}(n-1)!$ ▨

Rmk: This determines $\mu(\pi,\pi')$ for all $\pi,\pi' \in \Pi_n$ as follows:
If $\pi' = \{S_1, \ldots, S_\ell\}$ and
π refines block S_i into n_i blocks
then $[\pi,\pi'] \cong \Pi_{n_1} \times \Pi_{n_2} \times \cdots \times \Pi_{n_\ell}$
so $\mu(\pi,\pi') = (-1)^{n_1}(n_1-1)! \cdots (-1)^{n_\ell}(n_\ell-1)!$

12/1 Computing Möbius functions of lattices (§3.8, 3.9 Stanley)

Def'n For a lattice L, its Möbius algebra $A(L,\mathbb{C})$, over complex numbers $\mathbb{C}$, is $\mathbb{C}^L$ with $\mathbb{C}$-basis $\{f_x\}_{x\in L}$ that multiplies by the rule $f_x \cdot f_y = f_{x\wedge y}$.

Prop. for a finite lattice L, there is a (ring) isomorphism

$$A(L,\mathbb{C}) \xrightarrow{\varphi} \mathbb{C}^{|L|} := \{\underbrace{\mathbb{C}\times\mathbb{C}\times\cdots\times\mathbb{C}}_{|L| \text{ times}} \text{ w/ } \mathbb{C}\text{-basis } \{e_x\}_{x\in L}$$

$$f_y \mapsto \sum_{x\le y} e_x$$

that multiply as orthogonal idempotents $e_x^2 = e_x,\ e_x e_y = 0$ if $x \ne y\}$.

We have $\delta_y := \varphi^{-1}(e_y) = \sum_{x\le y}\mu(x,y) f_x$, so $f_y = \sum_{x\le y}\delta_x$.
Hence $\{\delta_y\}_{y\in L}$ are a $\mathbb{C}$-basis of orthogonal idempotents in $A(L,\mathbb{C})$.

Proof: φ is a $\mathbb{C}$-vector space iso. since its matrix is uniuppertriangular

$$\varphi = f_y\left\{\overset{e_x}{\boxed{\begin{matrix}1 & & *\\ & \ddots & \\ & & 1\end{matrix}}}\right.$$ for any linear ordering of L that extends $\le$.

Also can check $\varphi(f_y f_z) = \varphi(f_{y\wedge z}) = \sum_{x\le y\wedge z} e_x$

and $\varphi(f_y)\varphi(f_z) = \left(\sum_{x\le y} e_x\right)\left(\sum_{w\le z} e_w\right) = \sum_{\substack{x\le y,\\ w\le z}} e_x e_w = \sum_{\substack{x\le y,\\ x\le z}} e_x = \sum_{x\le y\wedge z} e_x$ ✓

The fact that $\varphi^{-1}(e_y) = \sum_{x\le y}\mu(x,y) f_y$ follows from $f_y = \sum_{x\le y}\varphi^{-1}(e_x)$ via Möbius inversion.

Cor (Weisner's Thm)

If $a \lneq \hat{1}$ in finite lattice L, then $\sum_{x: a\wedge x = \hat{0}} \mu(x,\hat{1}) = 0$.

(Dually, if $a \gneq \hat{0}$, then $\sum_{x: a \vee x = \hat{1}} \mu(\hat{0}, x) = 0$.)

Proof: Compute in 2 ways

$$\left(\sum_{b\le a}\delta_b\right)\delta_{\hat{1}} = f_a\,\delta_{\hat{1}} = f_a\cdot\left(\sum_{x\le\hat{1}}\mu(x,\hat{1})\cdot f_x\right)$$

$$\left(\sum_{b\le a}\delta_b\right)\delta_{\hat{1}} = 0, \text{ since } b\le a \Rightarrow b \ne \hat{1}$$

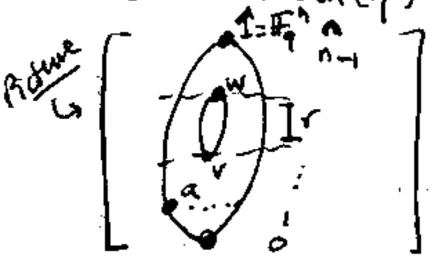
$$f_a\cdot\left(\sum_{x\le\hat{1}}\mu(x,\hat{1})\cdot f_x\right) = \sum_{x\in L}\mu(x,\hat{1})\, f_{a\wedge x}.$$

extract coeff of $f_{\hat{0}}$

$$0 = \sum_{x: a\wedge x=\hat{0}} \mu(x,\hat{1}).$$ ▨

Example of use of Weisner's Thm.!

Prop: In $\mathcal{L}_n(q)$ (vector space lattice), $\mu(\hat{0},\hat{1}) = (-1)^n q^{\binom{n}{2}}$, and hence $\mu(V,W) = (-1)^r q^{\binom{r}{2}}$ if $\dim(W/V) = r$.



Proof:

Pick a line a, and then

$$0 = \sum_{x: a\vee x = \hat{1}} \mu(\hat{0}, x)$$

$$\mu(\hat{0},\hat{1}) = -\sum_{x \lneq \hat{1},\, a\vee x = \hat{1}} \mu(\hat{0}, x)$$

← forces x to have $\dim = n-1$ since $\underbrace{\dim(x+a)}_{x\vee a} = \dim(x) + \dim(a) - \dim(x\cap a) \le \dim(x)+1$

count # of x of $\dim = n-1$ s.t. $a \not\le x$

$$= -\left(\left[{n \atop 1}\right]_q - \left[{n-1 \atop 1}\right]_q\right)\mu_{\mathcal{L}_{n-1}(q)}(\hat{0},\hat{1})$$

$$= -\left((1+q+\cdots+q^{n-1}) - (1+q+\cdots q^{n-2})\right)\cdot\mu_{\mathcal{L}_{n-1}(q)}(\hat{0},\hat{1})$$

$$= -q^{n-1}\mu_{\mathcal{L}_{n-1}(q)}(\hat{0},\hat{1}) \underset{\text{induction}}{=} (-1)^n q^{(n-1)+(n-2)+\cdots+2+1+0}$$

$$= (-1)^n q^{\binom{n}{2}}.$$ ▨

12/3

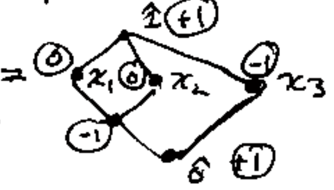
To compute μ for distr. lattice $J(P)$, let's use another thm:

Thm (Rota's Crosscut Thm) $\quad$ = elts. $x \lessdot \hat{1}$

In a finite lattice L, w/ coatoms $\{x_1, \dots, x_\ell\}$, we have
$$\mu(\hat{0}, \hat{1}) = \sum_{\substack{S \subseteq \{x_1, \dots, x_\ell\},\\ \wedge S = \hat{0}}} (-1)^{\#S}$$

In particular, $\mu(\hat{0}, \hat{1}) = 0$ if $\hat{0}$ is not a meet of coatoms.

e.g.

$L =$ 

$\boxed{\mu(\hat{0}, x)}$

S	$(-1)^{\#S}$
$\{1,3\}$	$+1$
$\{2,3\}$	$+1$
$\{1,2,3\}$	-1

$+1 = \mu(\hat{0}, \hat{1})$ ✓

Pf: In the Möbius algebra $A(L, \mathbb{C})$, compute in 2 ways:
$$\sum_{S \subseteq \{x_1, \dots, x_\ell\}} (-1)^{\#S} \prod_{i \in S} f_{x_i} = \prod_{i=1}^{\ell} (f_{\hat{1}} - f_{x_i}) = \prod_{i=1}^{\ell} \Big(\sum_{y \not\leq x_i} \delta_y \Big)$$

$$= \sum_{S \subseteq \{x_1, \dots, x_\ell\}} (-1)^{\#S} f_{\wedge S}$$

} extract coeff. of $f_{\hat{0}}$ $\downarrow$

Theorem ✓

$$= \sum_{\substack{(y_1, \dots, y_\ell):\\ y_i \not\leq x_i}} \delta_{y_1} \cdots \delta_{y_\ell} = \sum_{y \not\leq x_i \forall i} \delta_y = \delta_{\hat{1}}$$

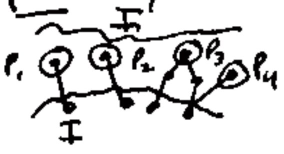
since every $y \neq \hat{1}$ below some x_i

■

Def'n An antichain $A \subseteq P$ is a subset of pairwise incomparable elts.

Cor In finite distr. lattice $L = J(P)$,
$$\mu(I, I') = \begin{cases} (-1)^{\#I' \setminus I} & \text{if } I' \setminus I \text{ is antichain in } P \\ 0 & \text{otherwise.} \end{cases}$$

Pf:



Check that coatoms of $[I, I']$ are $x_i = I' \setminus \{p_i\}$ for maximal $p_i \in I' \setminus I$.

So their meet $x_1 \wedge \cdots \wedge x_\ell = I' \setminus \{p_1, \dots, p_\ell\} = I$

$\Leftrightarrow$ every elt. of $I \setminus I'$ is max'l, i.e., $I \setminus I'$ is an antichain! ■

And... that's the end of the material for the course!

Congratulations! and... let me advertise

Math 274 - Combinatorics II - Spring 2022

We will continue the study/enumeration of discrete structures, with a new focus on symmetries! (a.k.a. algebra!)

Two main topics:

① Enumeration under group action:

⇝ How many ways are there to color the faces of a cube w/ 3 colors if we consider colorings the same if we can rotate the cube to get from one coloring to the other?

② Symmetric functions.

Consider polynomial: $P(x) = (x-a)(x-b)(x-c)$ w/ roots a, b, c

Expanding ... $P(x) = x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc$

the coefficients of $p(x)$ are themselves poly.'s in a, b, c, and invariant under permuting a, b, c: called symmetric polynomials

Symmetric polynomials have rich combinatorial structure!

See samuelfhopkins.com/classes/274.html for more info...