## Howard Math 273, HW # 1,

Fall 2023; Instructor: Sam Hopkins; Due: Friday, September 29th

1. (Stanley, EC1, #1.66) Let  $p_k(n)$  denote the number of partitions of n into exactly k parts. Give a **bijective** proof that

$$p_0(n) + p_1(n) + p_2(n) + \dots + p_k(n) = p_k(n+k)$$

Hint: Think about Young diagrams.

2. (Stanley, EC1, #1.5) Show that, for any  $k \ge 1$ ,

$$\sum_{n_1,\dots,n_k\geq 0} \min(n_1,\dots,n_k) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1-x_1)(1-x_2) \cdots (1-x_k) \cdot (1-x_1 x_2 \cdots x_k)},$$

where  $\min(n_1, \ldots, n_k)$  means the minimum of the integers  $n_1, \ldots, n_k$ .

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3. (Stanley, EC1, #1.26) Let  $\overline{c}(n, m)$  denote the number of compositions of n into parts of size at most m. Show that

$$\sum_{n \ge 0} \overline{c}(n,m) x^n = \frac{1-x}{1-2x+x^{m+1}}$$

4. Prove that, for any  $n \ge 0$ ,

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^{n}.$$

**Hint**: We discussed the generating function  $\sum_{n\geq 0} {\binom{2n}{n}} x^n$  of the central binomial coefficients. How can you use what we proved about this generating function to deduce the desired result?

5. Let ODD(n) denote the subset of permutations in the symmetric group  $S_n$  that have no cycles of even size. Prove that, for any  $n \ge 1$ ,

$$\sum_{\sigma \in \text{ODD}(n)} 2^{\#\text{cycles}(\sigma)} = 2 \cdot n!.$$

**Hint**: Recall that we showed

$$\sum_{n\geq 0} \left( \sum_{\sigma\in S_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots t_n^{c_n(\sigma)} \right) \frac{x^n}{n!} = e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + t_4 \frac{x^4}{4} + \cdots},$$

where  $c_k(\sigma)$  is the number of cycles of  $\sigma$  of size k. How can you use this generating function identity to deduce the desired result?