Howard Math 273, HW# 1,

Fall 2021; Instructor: Sam Hopkins; Due: Friday, October 1st

1. *(Stanley, EC1, #1.66)* Let $p_k(n)$ denote the number of partitions of n into k parts. Prove bijectively that
$$p_0(n) + p_1(n) + p_2(n) + \cdots + p_k(n) = p_k(n+k).$$

2. *(Stanley, EC1, #1.113)* Fix natural numbers k, n. Let $[n]$ denote the set $[n] := \{1, 2, \ldots, n\}$. Give a simple formula for the number of ordered k-tuples $(T_1, \ldots, T_k)$ of subsets of $[n]$ satisfying
   - $T_i \cap T_j = \varnothing$ for all $i \neq j$ (i.e., they are disjoint);
   - $\bigcup_{i=1}^{k} T_i = [n]$ (i.e., their union is the whole set $[n]$).

   (This is almost the same as saying that the T_i form a set partition of $[n]$, except that some of these sets may be empty, which we do not usually allow for set partitions.)

3. *(Stanley, EC1, #1.5)* Show that
$$\sum_{n_1,\ldots,n_k \geq 0} \min(n_1, \ldots, n_k) x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} = \frac{x_1 x_2 \cdots x_k}{(1-x_1)(1-x_2)\cdots(1-x_k)\cdot(1-x_1x_2\cdots x_k)}.$$

4. *(Stanley, EC1, #1.26)* Let $\overline{c}(n, m)$ denote the number of compositions of n into parts of size at most m. Show that
$$\sum_{n \geq 0} \overline{c}(n, m) x^n = \frac{1-x}{1-2x+x^{m+1}}.$$

5. Prove that, for any $n \geq 0$,
$$4^n = \sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k}. \tag{1}$$

   Hint: did we discuss the generating function $\sum_{n=0}^{\infty} \binom{2n}{n} x^n$ of the central binomial coefficients?
   Hard bonus problem, just to think about, not to do: prove (1) *bijectively* (see Stanley, EC1, #1.3(c)).

6. Let $n \geq 1$, and let $\mathrm{ODD}(n)$ denote the subset of permutations in the symmetric group $\mathfrak{S}_n$ with no cycles of even size. Prove that
$$\sum_{\sigma \in \mathrm{ODD}(n)} 2^{\#\mathrm{cycles}(\sigma)} = 2 \cdot n!. \tag{2}$$

   Hint: recall *Touchard's theorem*
$$\sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots t_n^{c_n(\sigma)} \right) x^n = e^{t_1 \frac{x}{1} + t_2 \frac{x^2}{2} + t_3 \frac{x^3}{3} + \cdots} = e^{\sum_{j=1}^{\infty} t_j \frac{x^j}{j}},$$
   where $c_k(\sigma)$ is the number of cycles of σ of size k.
   Hard bonus problem, just to think about, not to do: prove (2) *bijectively* (see §6.2 of M. Bona, "A Walk Through Combinatorics," 3rd ed., for the idea).