

Howard Math 273, HW# 3,

Fall 2023; Instructor: Sam Hopkins; Due: Friday, December 1st

1. The *complete bipartite graph* $K_{n,m}$ is the graph with vertex set $X \cup Y$ where $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$, and with edges $\{x_i, y_j\}$ for all $1 \leq i \leq n, 1 \leq j \leq m$ (but with no edges between the x 's, or between the y 's). Use the Matrix-Tree Theorem to show that the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$.

Hint: You can use the fact that for a matrix M in *block form* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have $\det(M) = \det(A - BD^{-1}C) \cdot \det(D)$ if D is invertible. Compare to: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

2. (Stanley, EC1, #4.69) Compute the number of closed walks of length ℓ in the complete bipartite graph $K_{n,m}$. Use this computation, together with the Transfer Matrix Method, to find the eigenvalues of the adjacency matrix of $K_{n,m}$.
3. (Stanley, EC1, #3.34) Recall that for a poset P , $\mathcal{J}(P)$ denotes the set of *order ideals* of P (i.e., subsets $I \subseteq P$ for which $q \in I$ and $p \leq q \in P$ implies $p \in I$). Find **all** finite posets P for which

$$\sum_{I \in \mathcal{J}(P)} x^{\#I} = (1+x)(1+x^2)(1+x+x^2).$$

Hint: How many order ideals must such a P have? How many elements must P have? How many *minimal* elements must it have? How many *maximal* elements must it have?

4. Let P be a finite poset. An *antichain* A of P is a subset $A \subseteq P$ of pairwise incomparable elements (i.e., for all $p, q \in A$, we have neither $p \leq q$ nor $q \leq p$). Let $\mathcal{A}(P)$ denote the set of antichains of P . Define a partial order \preceq on $\mathcal{A}(P)$ where $A \preceq A'$ means that for every $p \in A$ there is some $p' \in A'$ with $p \leq p'$. Show that $(\mathcal{A}(P), \preceq)$ is isomorphic to $(\mathcal{J}(P), \subseteq)$, the distributive lattice of order ideals of P ordered by containment.
5. (Stanley, EC1, #3.89) Let L be a finite lattice, with minimum element $\hat{0}$. Let $f_L(m)$ be the number of m -tuples $(t_1, \dots, t_m) \in L^m$ such that $t_1 \wedge t_2 \wedge \dots \wedge t_m = \hat{0}$. Use Möbius inversion to show that

$$f_L(m) = \sum_{t \in L} \mu(\hat{0}, t) \cdot (\#\{s \in L : s \geq t\})^m,$$

where μ is the Möbius function of L .

Hint: Define $f_L(m, t) := \#\{(t_1, \dots, t_m) \in L^m : t_1 \wedge t_2 \wedge \dots \wedge t_m = t\}$ for any $t \in L$ (so that $f_L(m) = f_L(m, \hat{0})$), and also define $g_L(m, t) := \#\{(t_1, \dots, t_m) \in L^m : t_1 \wedge t_2 \wedge \dots \wedge t_m \geq t\}$. How are these f and g related? Can you find a simpler expression for g ?