Howard Math 273, HW# 4,

Fall 2026; Instructor: Sam Hopkins; Due: Wednesday, November 25th

1. The q-binomial coefficient satisfies $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \mathcal{W}_{n,k}} q^{\mathrm{inv}(w)}$, where $\mathcal{W}_{n,k}$ is the set of words that are rearrangements of $(n-k)$ 0's and k 1's, and $\mathrm{inv}(w)$ is the number of inversions of w.

   Suppose $n = 2m$ is even. Prove that $\begin{bmatrix} n \\ k \end{bmatrix}_{q:=-1}$ (the evaluation of the q-binomial at $q = -1$) is equal to $\#\mathcal{P}_{n,k}$, where $\mathcal{P}_{n,k}$ is the subset of words $w = w_1 w_2 \dots w_n \in \mathcal{W}_{n,k}$ that are *palindromes*, i.e., which satisfy $w_i = w_{n+1-i}$ for all i. Do this by defining a **sign-reversing involution**. That is, define an involution $\tau \colon \mathcal{W}_{n,k} \to \mathcal{W}_{n,k}$ satisfying:

   - $\mathrm{inv}(w)$ and $\mathrm{inv}(\tau(w))$ have opposite parity for all $w \in \mathcal{W}_{n,k}$ with $\tau(w) \neq w$;
   - $\mathrm{inv}(w)$ is even for all $w \in \mathcal{W}_{n,k}$ with $\tau(w) = w$;
   - $\#\{w \in \mathcal{W}_{n,k} \colon \tau(w) = w\} = \#\mathcal{P}_{n,k}$.

2. The *complete bipartite graph* $K_{n,m}$ is the graph with vertex set $X \cup Y$ where $X = \{x_1, ..., x_n\}$ and $Y = \{y_1, ..., y_m\}$, and with edges $\{x_i, y_j\}$ for all $1 \leq i \leq n, 1 \leq j \leq m$ (but with no edges between the x's, or between the y's). Use the Matrix-Tree Theorem to show that the number of spanning trees of $K_{n,m}$ is $n^{m-1}m^{n-1}$.

   Hint: You can use the fact that for a matrix M in *block form* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have $\det(M) = \det(A - BD^{-1}C) \cdot \det(D)$ if D is invertible. Compare to: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$.

3. *(Stanley, EC1, #3.34)* Recall that for a poset P, $\mathcal{J}(P)$ denotes the set of *order ideals* of P (i.e., subsets $I \subseteq P$ for which $q \in I$ and $p \leq q \in P$ implies $p \in I$). Find **all** finite posets P for which

$$\sum_{I \in \mathcal{J}(P)} x^{\#I} = (1+x)(1+x^2)(1+x+x^2).$$

   Hint: How many order ideals must such a P have? How many elements must P have? How many *minimal* elements must it have? How many *maximal* elements must it have?

4. Let G be a graph with vertex set $[n] := \{1, 2, \dots, n\}$. A set partition $\pi = \{\pi_1, \pi_2, \dots, \pi_k\}$ of $[n]$ is called *G-connected* if the restriction of G to each block π_i is connected. Define L_G, the *bond lattice* of G, to be the poset whose elements are the G-connected partitions of $[n]$, with $\pi \leq \pi'$ if each block of π is a subset of some block of π'. Define $\chi_G(x) := \sum_{\pi \in L_G} \mu(\hat{0}, \pi) x^{\#\pi}$, where $\hat{0} = \{\{1\}, \dots, \{n\}\}$ (the all singletons partition) is the minimum of L_G, and μ is the Möbius function of L_G. Prove that, for every positive integer x, $\chi_G(x)$ is the number of ways to color the vertices of G with colors $\{1, 2, \dots, x\}$ such that adjacent vertices are colored different colors. (This $\chi_G(x)$ is called the *chromatic polynomial* of the graph G.)

   Hint: For $\pi \in L_G$, define $g(\pi, x)$ to be the number of vertex colorings with x colors so that all blocks in π are monochromatic, and $f(\pi, x)$ to be the number which additionally have adjacent blocks colored differently. Notice that $\chi_G(x) = f(\hat{0}, x)$. Explain how these $f(\pi, x)$ and $g(\pi, x)$ are related, give a simple formula for $g(\pi, x)$, and then apply Möbius inversion.